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# POSITIVITY CONSTRAINTS ON LORENTZ-BREAKING EFTs

Work in progress with Oliver Janssen and Leonardo Senatore

2207.XXXXX + ...

COPERNICUS Webinar

# POSITIVITY : LI CASE

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06

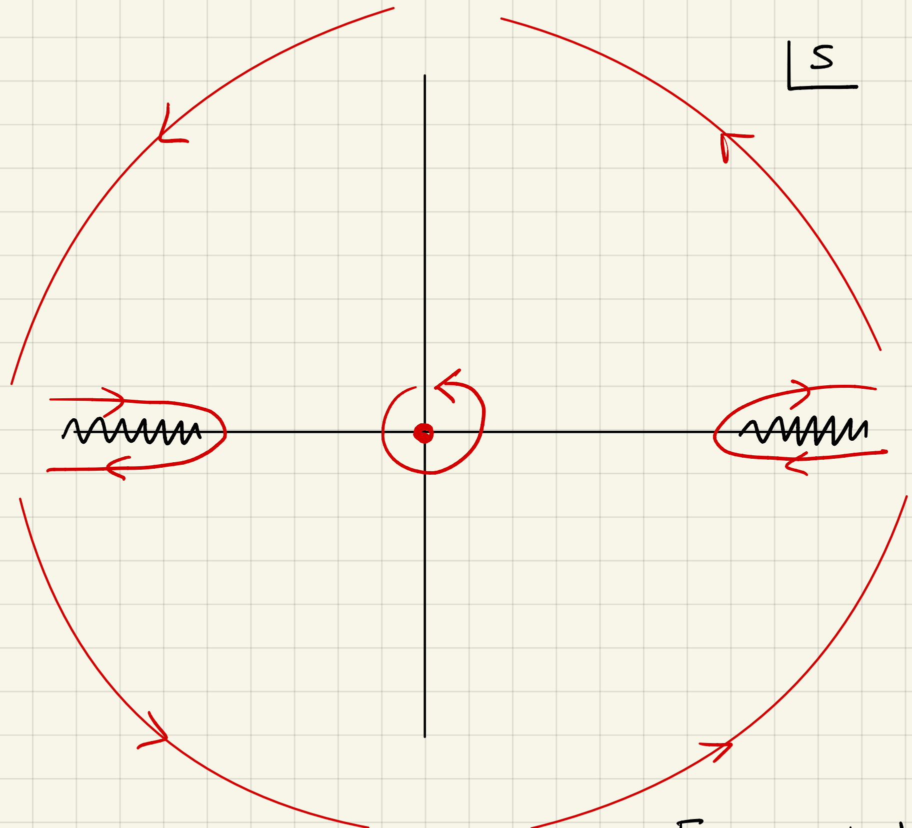
$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 + \frac{c_3}{\Lambda^4}(\partial\pi)^4 \qquad c_3 \geq 0$$

Forward amplitude $\mathcal{A}(s) = \mathcal{M}(s; t\to 0)$ In EFT : $\mathcal{A}(s) = c_3 \frac{s^2}{\Lambda^4}$

$\mathcal{A}(s)$ analytic apart from cut on $\mathbb{R}$

$\mathcal{A}(s) = \mathcal{A}^*(-s^*)$ crossing

$$\oint \frac{ds}{2\pi i} \frac{\mathcal{A}(s)}{s^3} = \frac{c_3}{\Lambda^4}$$




This can be rewritten as a UV integral:

$$\frac{c_3}{\Lambda^4} = \frac{2}{\pi}\int ds \, \frac{s\,\sigma(s)}{s^3} > 0$$

Froissart bound:

$$\mathcal{A}(s) < s \log^2 s$$

# THE SAME FOR NON-LI THEORIES ?

## MOTIVATIONS :

We are interested in theories in which Lorentz is spontaneously broken

- Cosmology. In particular Inflation and Dark Energy / Modification of Gravity

  In general the EFT is defined on a ~~LI~~ background, cannot be "extrapolated" from LI background

  We are particularly interested in weird theories. Are they consistent?

- Condensed Matter . Here clearly there is no hope in starting LI

- QFT at finite T or charge Q

Baumann, Green, Lee, Porto '15
Grall, Melville '21

# ... JUST DO THE SAME?

$$\mathcal{L} = \dot{\pi}^2 - c_s^2(\partial_i\pi)^2 + \frac{\alpha_1}{\Lambda^2}\dot{\pi}^3 + \frac{\alpha_2}{\Lambda^2}\dot{\pi}(\partial_i\pi)^2 + \dots$$

One can look at the scattering of $\pi$'s.


In a LI theory this is well-defined at arbitrary energy. (Calculable in EFT only at low energy)

In a ~~LI~~ theory I cannot scatter $\pi$ at high energy. These are not good asymptotic states

Scatter a superfluid wave at 10 TeV?

What is the object whose analyticity we want to use?

- Find analytic function whose low E limit gives $\mathcal{A}_{EFT}$
- Find analogue of Froissart bound

?

# TRY SOMETHING MORE BASIC

1. Assume to flow to a CFT in UV (general ?)

2. UV behaviour of $\langle J^\mu J^\nu \rangle$ and $\langle T^{\mu\nu} T^{\alpha\beta} \rangle$ is known

$$\langle J^\mu(p)\, J^\nu(-p) \rangle = C_J \frac{p^2 g^{\mu\nu} - p^\mu p^\nu}{p^{4-d}}$$

3. At low energy EFT (is not conformal and) breaks LI spontaneously

4. Analiticity and unitarity of $\langle JJ \rangle$ and $\langle TT \rangle$, together with UV limit above implies positivity properties in EFT

# Conformal Superfluid

- Superfluid: $\int d^4x \sqrt{-g}\ P(X) \qquad X \equiv -\partial_\mu\phi\partial^\mu\phi$

  Expanded around $\phi = c \cdot t$ it describes a perfect fluid w/o vortices

  $$T_{\mu\nu} = 2P'(X)\,\partial_\mu\phi\partial_\nu\phi + P(X)\,g_{\mu\nu} \qquad \rho = 2P'X - P \qquad p = P \qquad u_\mu = \frac{\partial_\mu\phi}{\sqrt{-X}}$$

  E.g. $w = \frac{1}{3} \qquad P(X) = X^2$

  Poincaré $\times\ U(1) \longrightarrow$ Spacetime translations $\times$ Rotations ($\times$ Shift)

- To be concrete focus on a CFT at finite $\mu$. $U(1)$ spont. broken
  At $E \ll \mu$ the system is described by an EFT

  $$\chi(t;\vec{x}) = \mu t + \pi(t;\vec{x})$$

  $SO(d;2) \times U(1) \longrightarrow$ Spacetime translations $\times$ Rotations ($\times$ Shift)

# CFTs AT LARGE CHARGE

Hellerman et al '15
Rattazzi et al '15 + ...

Action by coset construction or out of the metric $\hat{g}_{\mu\nu} \equiv g_{\mu\nu} |g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi|$

$$S^{(1)} = \frac{c_1}{6} \int d^d x \sqrt{-\hat{g}} = \frac{c_1}{6} \int d^d x \sqrt{-g} \, |\partial \chi|^d$$

$$S^{(2)} = \int d^d x \sqrt{-\hat{g}} \left( -c_2 \hat{R} + c_3 \hat{R}^{\mu\nu} \hat{\partial}_\mu \chi \hat{\partial}_\nu \chi \right) = c_2 \int d^d x \sqrt{-g} \, |\partial \chi|^{d-2} \left( R + (d-1)(d-2) \frac{\nabla_\mu |\partial \chi| \nabla^\mu |\partial \chi|}{|\partial \chi|^2} \right) + c_3 \ldots$$

Expansion $\dfrac{\partial}{\mu}$

You want to study: $\langle O_{-Q}(x_{out}) \ldots O_Q(x_{in}) \rangle$ in $\mathbb{R}^d$

possibly other operators with finite charge

By operator - state correspondence we are in a state with large $Q$ on $S^{d-1} \times \mathbb{R}$

For large $Q$ one has an EFT with a single Goldstone and CFT results can be obtained as an expansion in $1/Q$

On cylinder this gives $\dfrac{1}{\mu R} \sim \dfrac{1}{Q^{1/2}}$

1-loop correction

E.g. In 3d: $$\Delta_Q = \frac{2}{3} \frac{Q^{3/2}}{\sqrt{2\pi c_1}} + 8\pi c_2 \sqrt{\frac{Q}{2\pi c_1}} - 0.0937256 + \mathcal{O}\left(Q^{-1/2}\right)$$

# Constraints from Current

Focus on 3d

$$\mathcal{L} = \frac{c_1}{6}\sqrt{-g}\,|\partial\chi|^3 - c_2\sqrt{-\hat{g}}\,\hat{R} + c_3\sqrt{-\hat{g}}\,\hat{R}^{\mu\nu}\hat{\partial}_\mu\chi\,\hat{\partial}_\nu\chi - \frac{b}{4}\frac{F_{\mu\nu}F^{\mu\nu}}{|\nabla\chi|} - \frac{d}{2}\frac{F_{\alpha\mu}F^{\mu}{}_{\beta}}{|\nabla\chi|}\partial^\alpha\chi\,\partial^\beta\chi$$

Can we get constraints on $c_{1;2;3}$ $b$ and $d$ using $\langle JJ\rangle$ ?

$$\langle J^\mu(-k)\,J^\nu(k)\rangle = -iA\left(k^\mu k^\nu - \eta^{\mu\nu}k^2\right) + iB\left(k^i k^j - \delta^{ij}\vec{k}^2\right)$$

$$A = -\frac{\mu c_1}{2\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)} + \frac{c_2}{\mu}\frac{\left(\omega^2-\boldsymbol{k}^2\right)\boldsymbol{k}^2}{\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)^2} - \frac{c_3}{\mu}\frac{\omega^2\boldsymbol{k}^2}{\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)^2} + \frac{b}{\mu} + \frac{d}{\mu},$$

$$B = -\frac{\mu c_1}{4\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)} - \frac{c_2}{\mu}\frac{\left(\omega^2-\boldsymbol{k}^2\right)^2}{\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)^2} + \frac{c_3}{\mu}\frac{\omega^2(\omega^2-\boldsymbol{k}^2)}{\left(\omega^2 - c_s^2\boldsymbol{k}^2\right)^2} + \frac{d}{\mu}.$$

# Current Analiticity

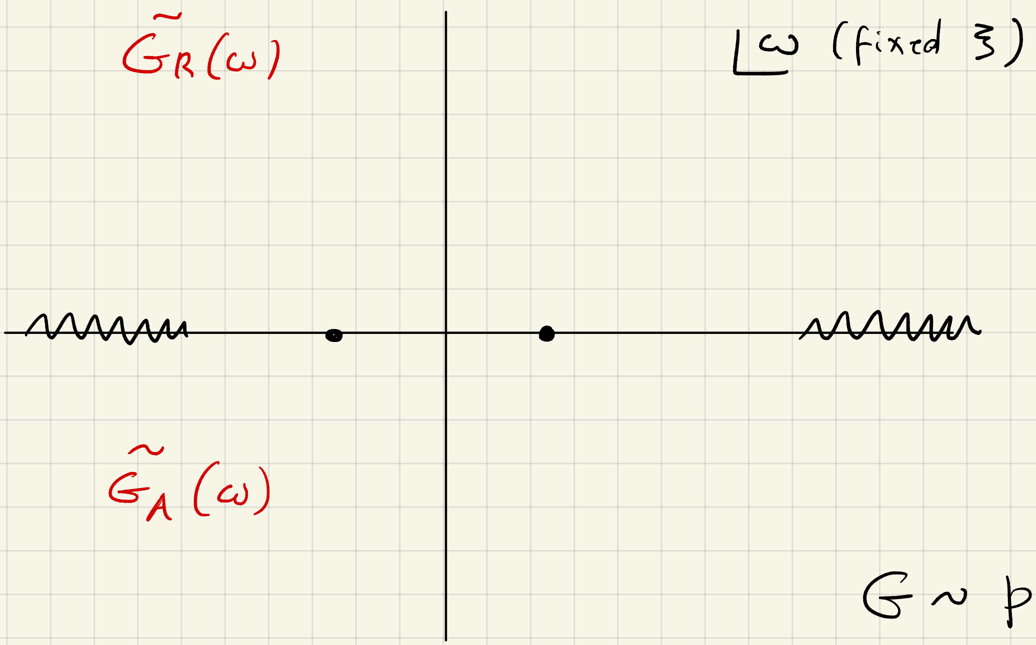
I was sloppy about dealing with the poles. $G_R^{\mu\nu}(x-y) = \Theta(x^0-y^0)\langle 0|[J^\mu(x)J^\nu(y)]|0\rangle$

(Remember Kramers - Kronig) $\tilde{G}_R^{\mu\nu}(\omega;\vec{p})$

$\tilde{G}_R(\omega;\vec{p})$ is analytic if $\mathrm{Im}\, p^\mu \in$ I can take $\vec{p} = \xi\omega\hat{e}$, $|\xi|<1$

$\tilde{G}_A$ has same properties but in lower half plane

$$\tilde{G}_R^{\mu\nu}(\omega) - \tilde{G}_A^{\mu\nu}(\omega) = (2\pi)^d \sum_n \delta^{(d)}(p-P_n)\overbrace{\langle 0|J^\mu(0)|P_n\rangle\langle P_n|J^\nu(0)|0\rangle}^{\geq 0 \text{ matrix}} - \delta^{(d)}(p+P_n)\langle 0|J^\nu(0)|P_n\rangle\langle P_n|J^\mu(0)|0\rangle$$




$\tilde{G}_R$ and $\tilde{G}_A$ are continuation of each other

$G \sim p$ at large $|p|$

$$\frac{1}{2\pi i}\oint_{\text{origin}} \frac{G^{\mu\nu}(\omega; \vec{p} = \hat{e}\,\xi\,\omega)\, V_\nu V_\mu}{\omega^3} = \frac{1}{2\pi i}\int \frac{G^{\mu\nu}(\omega; \vec{p} = \hat{e}\,\xi\,\omega)\, V_\nu V_\mu}{\omega^3} \geq 0$$

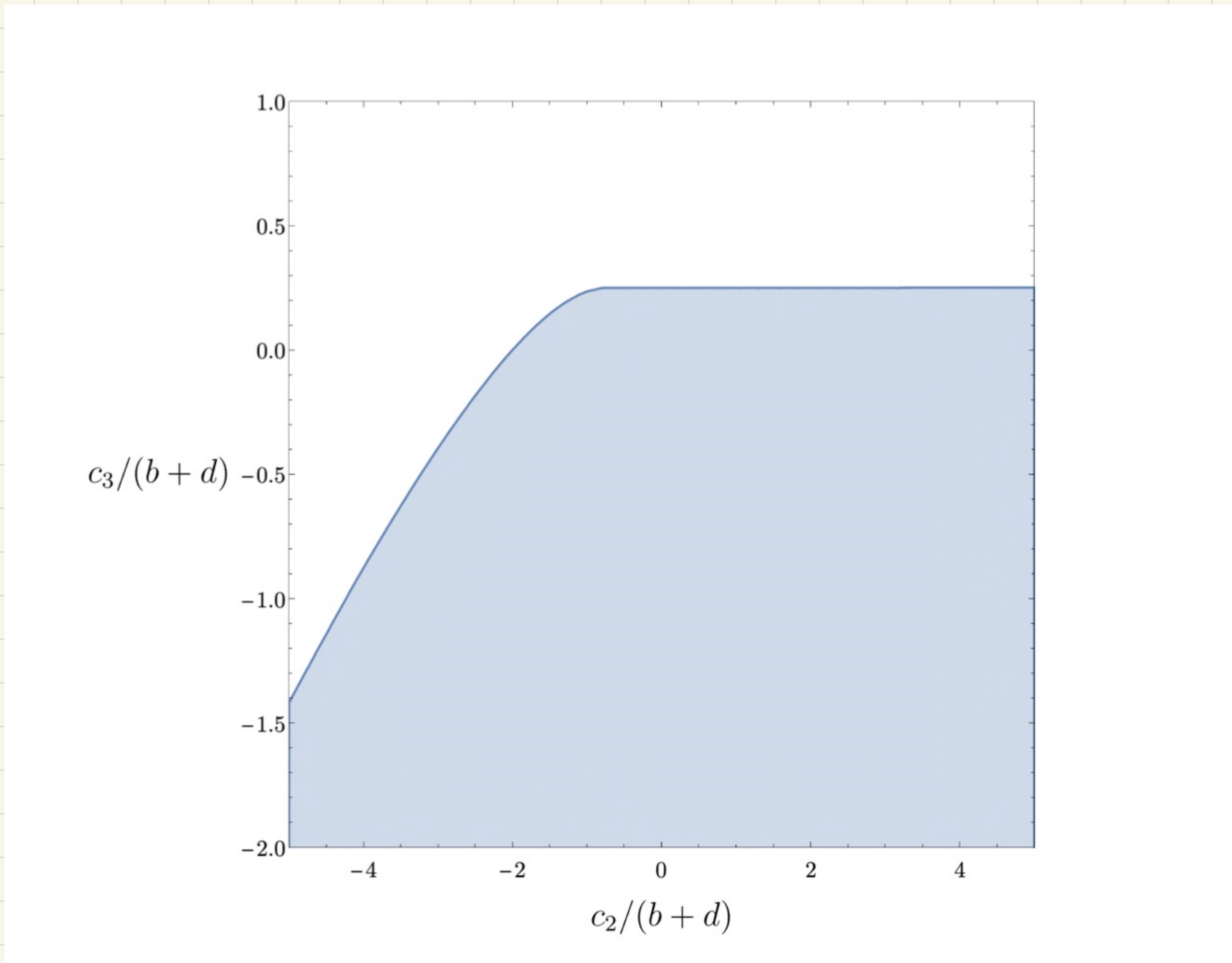
For any $\xi < 1$ and any $V^\mu$

$$\frac{c_2}{\mu}\frac{\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2 - \frac{c_3}{\mu}\frac{\xi^2}{(1-\xi^2/2)^2}\beta^2 + \frac{b}{\mu}(\beta^2+\gamma^2) + \frac{d}{\mu}\left(\beta^2 + \frac{\gamma^2}{1-\xi^2}\right) \geq 0$$

$\beta$ and $\gamma$ parametrize $V^\mu$

$b \geq 0$

$b + d \geq 0$



# $T_{\mu\nu}$ Positivity

Given its behaviour at $\infty$, I have to divide by $\omega^5$. So I will get a constraint on higher order operators

$$\int d^3x \sqrt{-\hat{g}} \left[ \frac{c_1}{6} - c_2 \hat{R} + c_3 \hat{R}^{00} + c_4 \hat{R}^2 + c_5 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + c_7 \hat{R}^{0}{}_{\mu} \hat{R}^{\mu 0} + \dots \right]$$

higher order in $\partial$ or perturbations

Schematically:

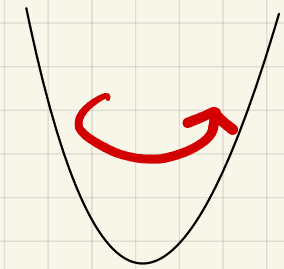
$$\frac{1}{2\pi i} \oint \frac{\langle T^{\mu\nu}(\omega; \hat{e}\,\xi\omega)\, T^{\alpha\beta}(\omega; \hat{e}\,\xi\omega)\rangle A_{\mu\nu} A_{\alpha\beta}}{\omega^5} = \int_{\supset\,\subset} [\text{SAME}] \geq 0$$

$$c_5 \geq 0 \qquad c_7 \geq 0 \qquad 4c_4 + 2c_5 + c_7 \geq 4\,\frac{(c_2+c_3)^2}{c_1}$$

# UV CHECKS

$$\mathcal{L}_{UV} = -|\partial\phi|^2 - \lambda|\phi|^6 - \frac{R}{8}|\phi|^2$$

At finite $\mu$: spinning



Conformal at tree-level

Integrate out radial direction $\phi = \rho e^{i\theta}$

$$\mathcal{L}_{EFT} = \frac{2}{3\sqrt{2\lambda}}|\partial\theta|^3 - \frac{1}{8\sqrt{3\lambda}}\left(2\frac{(\partial|\partial\theta|)^2}{|\partial\theta|} + |\partial\theta| R\right) + \dots$$

$$c_1 = \frac{4}{\sqrt{3\lambda}} \qquad c_2 = \frac{1}{8\sqrt{3\lambda}} \qquad c_4 = \frac{1}{256\sqrt{3\lambda}}$$

OK bounds. Saturates TT-bound

One can add another real scalar

$$\mathcal{L}_{UV} = \dots - (\partial\varphi)^2 - \lambda_2\varphi^6 - \beta_1|\phi|^2\varphi^4 - \beta_2|\phi|^4\varphi^2 - \frac{R}{8}\varphi^2$$

The bounds are non-trivially satisfied for all $\lambda, \lambda_2, \beta_1, \beta_2$

# RELATIVISTIC KRAMERS-KRONIG

Melrose Stoneham '76

Real and Imaginary part of index of refraction are related in a medium as a consequence of causality

$$\alpha^{\mu\nu}(\omega;\vec{k}_0) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\zeta}{\zeta-\omega}\,\alpha^{\mu\nu}(\zeta;\vec{k}_0)$$

This is just a consequence of causality : $\Theta(t)$

$$\alpha^{\mu\nu}(\omega;\vec{k}_0+\vec{\xi}\omega) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\zeta}{\zeta-\omega}\,\alpha^{\mu\nu}(\zeta;\vec{k}_0+\zeta\vec{\xi})$$

$|\vec{\xi}|<1$

Relativistic version

- The integral may not converge and we need to take $\frac{d^n}{d\omega^n}$ on both sides
- Before we chose $\vec{k}=\vec{\xi}\omega$ but $\vec{k}=\vec{k}_0+\vec{\xi}\omega$ is also ok. However it turns out it does not change bounds (it connects LHS and RHS by $\left(\frac{k_0}{\mu}\right)^{\#}$)
- Straightforward to prove it is equivalent to what we did before

# WHAT ABOUT LOOPS ?

See Bellazzini, Mirò, Rattazzi, Riembau, Riva '20

EFT loops complicate things because $c_i$ runs with energy and a cut opens on all real axis


One can deal with this but in our particular example there is no running (in 3d only)

Schematically:

$$\int d^3x \left[ \dot{\pi}_c^{\,2} - \frac{1}{2}(\partial_i \pi_c)^2 \right] + \frac{c_1}{\mu^{3/2}} \dot{\pi}_c^{\,3} + \frac{c_1}{\mu^3} \dot{\pi}_c^{\,4} + \dots \frac{c_2}{\mu^2} \partial^2\pi_c \, \partial^2 \pi_c + \frac{c_2}{\mu^{7/2}} (\partial^2 \pi_c)^2 \partial \pi_c$$

$$+ \frac{c_4}{\mu^4} \partial^3 \pi_c \, \partial^3 \pi_c + \dots$$

No way to combine interactions to get log running for $c_2$ and $c_4$

# Conclusions and Future

- Robust constraints on ~~LI~~ EFT
- Only on subleading operators ("more irrelevant than CFT"), which start quadratic

## To Do List:

- Explicit checks at loop order
- EFT coefficients of critical $O(N)$ at large $N$
- 4D. From $\langle TT \rangle$ one gets bounds on $\partial^6$ operators in EFT!
- Real systems: CM? Full set of constraints for light in a medium
- Finite $T$? (Delacretaz)
- Higher order $\langle JJJJ \rangle$?
- SOMETHING MORE GENERAL?