Towards a theory of nonlinear gravitational waves: a systematic approach to nonlinear gravitational perturbations in vacuum

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Problem

To solve vacuum Einstein equations

$$R_{\mu\nu}[g] - \kappa \frac{d}{\ell^2} g_{\mu\nu} = 0, \quad \kappa = 0, +1, -1, \quad \Lambda = \kappa \frac{d(d-1)}{2\ell^2},$$

in the form of a perturbation of some known exact solution $\bar{g}_{\mu\nu}$, i.e.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

Once we expand

$$\delta g_{\mu\nu} = \sum_{1 \leq i} {}^{(i)}h_{\mu\nu}\, \varepsilon^i$$

we describe linear ($i = 1$) and nonlinear ($i > 1$) gravitational waves

(examples: 1. asymptotically AdS time-periodic solutions, 2. what is the end-state of a generic perturbation of a Schwarzschild black hole?, ...)

Remark: we assume some symmetry of $\bar{g}_{\mu\nu}$ allowing for harmonic analysis in this symmetry subspace - it defines the modes (and polarizations) of **linear** gravitational waves; this talk: perturbations of Schwarzschild type (Kottler) solutions (spherical symmetry).

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

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(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

VOLUME 24, NUMBER 13 PHYSICAL REVIEW LETTERS 30 MARCH 1970

EFFECTIVE POTENTIAL FOR EVEN-PARITY REGGE-WHEELER GRAVITATIONAL PERTURBATION EQUATIONS*

Frank J. Zerilli

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(Received 29 January 1970)

The Schrödinger-type equation for odd-parity perturbations on a background geometry has been extended to the even-parity perturbations. This should greatly simplify the analysis for calculations of gravitational radiation from stars and from objects falling into black holes.

PHYSICAL REVIEW D, VOLUME 61, 044006

Gauge invariant formalism for second order perturbations of Schwarzschild spacetimes

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(Received 1 September 1999; published 24 January 2000)

The ''close limit,'' a method based on perturbations of Schwarzschild spacetime, has proved to be a very useful tool for finding approximate solutions to models of black hole collisions. Calculations carried out with second order perturbation theory have been shown to give the limits of applicability of the method without the need for comparison with numerical relativity results. Those second order calculations have been carried out in a fixed coordinate gauge, a method that entails conceptual and computational difficulties. Here we demonstrate a gauge invariant approach to such calculations. For a specific set of models (requiring head on collisions and quadrupole dominance of both the first and second order perturbations), we give a self-contained gauge invariant formalism. Specifically, we give (i) wave equations and sources for first and second order gauge invariant wave functions, (ii) the prescription for finding Cauchy data for those equations from initial values of the first and second fundamental forms on an initial hypersurface, and (iii) the formula for computing the gravitational wave power from the evolved first and second order wave functions.

PHYSICAL REVIEW D **74,** 044039 (2006)

Second- and higher-order perturbations of a spherical spacetime

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(Received 5 May 2006; published 30 August 2006)

The Gerlach and Sengupta (GS) formalism of coordinate-invariant, first-order, spherical and non-spherical perturbations around an arbitrary spherical spacetime is generalized to higher orders, focusing on second-order perturbation theory. The GS harmonics are generalized to an arbitrary number of indices on the unit sphere and a formula is given for their products. The formalism is optimized for its implementation in a computer-algebra system, something that becomes essential in practice given the size and complexity of the equations. All evolution equations for the second-order perturbations, as well as the conservation equations for the energy-momentum tensor at this perturbation order, are given in covariant form, in Regge-Wheeler gauge.

Outcome

In $3+1$:

- The problem of solving the system of **10** coupled linear inhomogeneous PDEs of mixed (**hyperbolic and elliptic**) type resulting from perturbation expansion of the Einstein equation is reduced, at each perturbation order, to solving **only 2** scalar wave (**hyperbolic**) equations for two **master scalar variables** and **some linear algebra**

In $d+1$, $d \geq 4$: there exist three (instead of two) **master scalar variables**

Perturbations in vacuum - general setup

Consider $R_{\mu\nu} - \kappa \frac{d}{\ell^2} g_{\mu\nu} = 0$ with $\kappa = 0, +1, -1$ and $\Lambda = \kappa \frac{d(d-1)}{2\ell^2}$.
Let $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

Now in Einstein equations $\delta R_{\mu\nu} - \kappa \frac{d}{\ell^2} \delta g_{\mu\nu} = 0$ expand $\delta g_{\mu\nu} = \sum_i \varepsilon^i h^{(i)}_{\mu\nu}$ itself and get the hierarchy of perturbative Einstein equations (expression for $\delta R_{\mu\nu}$ contains all powers of $\delta g_{\mu\nu}$):

$$\Delta_L h^{(i)}_{\mu\nu} = S^{(i)}_{\mu\nu}$$

Thus, we trade **nonlinearities** of Einstein equations for an **infinite system of linear inhomogeneous equations** (the sources $S^{(i)}_{\mu\nu}$ constructed from metric perturbations $h^{(j)}_{\mu\nu}$, with $j < i$). To solve it one needs:

1. a general solution of a principal (homogeneous) part
2. a particular solution of inhomogeneous part

Spherical symmetry & Regge-Wheeler decomposition ('57)

transformation of tensor components under rotations

rotations -transformation of angular variables preserving $(\gamma_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$, $(\varepsilon_{ab}) = \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)

$$T_{\alpha\beta} = \begin{pmatrix} S & S & V \\ S & S & V \\ V & V & T \end{pmatrix}$$

$$\begin{aligned}
S_{\ell m} &= Y_{\ell m}(\theta,\phi), \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)} \\
\left(\overset{1}{V}_{\ell m}\right)_a &= (S_{\ell m})_{;a}\,, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)} \\
\left(\overset{2}{V}_{\ell m}\right)_a &= \varepsilon_{ab}\gamma^{bc}\,(S_{\ell m})_{;c}\,, \quad \text{parity } (-1)^{\ell+1} \text{ (axial(or vector or odd) perturbation)} \\
\left(\overset{1}{T}_{\ell m}\right)_{ab} &= (S_{\ell m})_{;a;b}\,, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)} \\
\left(\overset{2}{T}_{\ell m}\right)_{ab} &= \gamma_{ab}S_{\ell m}, \quad \text{parity } (-1)^{\ell} \text{ (polar(or scalar or even) perturbation)} \\
\left(\overset{3}{T}_{\ell m}\right)_{ab} &= \varepsilon_{(ac}\gamma^{cd}\,(S_{\ell m})_{;d;b)}\,, \quad \text{parity } (-1)^{\ell+1} \text{ (axial(or vector or odd) perturbation)}
\end{aligned}$$

General approach to gravitational perturbations

1. At each perturbation order there exist two (for each gravitational mode) **masters scalar variables** satisfying inhomogeneous linear wave equations with uniquely defined potentials, i.e. there are only two polarization states in gravitational waves (cf. [Regge&Wheeler, 57], [Zerilli, 70] at linear order)
2. At each order gauge invariant components of metric perturbations (like Regge-Wheeler gauge invariant variables) are uniquely given in terms of master scalar variables and their derivatives (and some source functions at nonlinear orders) (cf. [Mukohyama, 00] at linear order, [Brizuela et al., 09] at second order)
3. These relations can be inverted for scalar master variables to be given in terms of RW type gauge invariant variables to provide the initial data and scalar sources for the scalar wave equations for master scalar variables (cf. [Moncrief, 74] at linear order, [Garat&Price, 00], [Brizuela et al., 09] at second order)

A few general remarks

1. Identities for the sources $S^{(i)}_{\mu\nu}$ comming from $\bar{\nabla}\left(\Delta_L h^{(i)}_{\ell\,\mu\nu} - S^{(i)}_{\ell\,\mu\nu}\right) = 0$. They are **crucial** for the consistency at higher orders
2. Gauge issues can become a nuisance [Bruni et al., 97] - we do not use fully gauge invariant approach to higher orders of perturbation expansion (cf. [Garat&Price, 00], [Brizuela et al., 09]).
   Gauge invariance in RW sense seems sufficient:
   $$x^{\mu} \to x^{\mu} + \zeta^{\mu}, \qquad \delta g_{\mu\nu} \to \delta g_{\mu\nu} + \mathscr{L}_{\zeta}\bar{g}_{\mu\nu} + \mathscr{O}\left(\zeta^2\right)$$
3. We use multipole expansion. At nonlinear orders of perturbation expansion the $\ell = 0, 1$ parts need special treatment
4. We limit ourselves to axial symmetry (stepping beyond axial symmetry is a technical, not a conceptual issue). Then we can limit ourselves to polar perturbations only (including axial perturbations to the scheme is straightforward)
5. We illustrate our approach on concrete examples in given coordinate systems
6. Including matter - postponed to the future work

Polar perturbations at axial symmetry (on concrete example)

Schwarzschild in static coordinates:

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2 d\Omega_2^2\,, \quad A = 1 - \kappa r^2/\ell^2 - 2M/r$$

$$h^{(i)}_{\alpha\beta} = \begin{pmatrix} h^{(i)}_{tt} & h^{(i)}_{tr} & h^{(i)}_{t\theta} & 0 \\ h^{(i)}_{tr} & h^{(i)}_{rr} & h^{(i)}_{r\theta} & 0 \\ h^{(i)}_{t\theta} & h^{(i)}_{r\theta} & h^{(i)}_{\theta\theta} & 0 \\ 0 & 0 & 0 & h^{(i)}_{\phi\phi} \end{pmatrix},$$

the sources $S^{(i)}_{\mu\nu}$ and perturbative Einstein equations $E^{(i)}_{\mu\nu}$ expanded into multipoles:

$$h^{(i)}_{tt}(t,r,\theta) = \sum_\ell h^{(i)}_{\ell\,tt}(t,r)P_\ell(\cos\theta)$$

$$h^{(i)}_{t\theta}(t,r,\theta) = \sum_\ell h^{(i)}_{\ell\,t\theta}(t,r)\partial_\theta P_\ell(\cos\theta)$$

RW gauge: only $h^{(i)}_{\ell\,tt}$, $h^{(i)}_{\ell\,rr}$, $h^{(i)}_{\ell\,tr}$, $h^{(i)}_{\ell\,+} = \left(h^{(i)}_{\ell\,\theta\theta} + h^{(i)}_{\ell\,\phi\phi}\,/\,\sin^2\theta\right)/2$ non zero, or out of **seven** polar metric components **four** RW gauge invariant functions $f^{(i)}_{\ell\,tt}, f^{(i)}_{\ell\,rr}, f^{(i)}_{\ell\,tr}, f^{(i)}_{\ell\,+}$ can be constructed $\left(h^{(i)}_{\ell\,tt} = f^{(i)}_{\ell\,tt} + 2\partial_t\zeta^{(i)}_{\ell\,t} - AA'\,\zeta^{(i)}_{\ell\,r}\right)$

here $\zeta^{(j)}_{\ell\,t}(t,r)$, $\zeta^{(j)}_{\ell\,r}(t,r)$, $\zeta^{(j)}_{\ell\,\theta}(t,r)$ define the j-th order polar gauge vector

$f^{(i)}_{\ell\, tt}(t,r), f^{(i)}_{\ell\, rr}(t,r), f^{(i)}_{\ell\, tr}(t,r)$ and $f^{(i)}_{\ell\, +}(t,r)$ are Regge-Wheeler (gauge invariant) variables

$\zeta^{(j)}_{\ell\, t}(t,r)$, $\zeta^{(j)}_{\ell\, r}(t,r)$, $\zeta^{(j)}_{\ell\, \theta}(t,r)$ define the j-th order polar gauge vector

$\zeta^{(j)}_{\alpha} = \sum_\ell \left(\zeta^{(j)}_{\ell\, t} P_\ell(\cos\theta),\, \zeta^{(j)}_{\ell\, r} P_\ell(\cos\theta),\, \zeta^{(j)}_{\ell\, \theta} \partial_\theta P_\ell(\cos\theta),\, 0 \right)$

and the corresponding gauge transformation $x^\mu \longrightarrow x^\mu + \varepsilon^j \zeta^{(j)\mu}$

$$\sum_{1\leq i} \varepsilon^i h^{(i)}_{\mu\nu} \to \sum_{1\leq i} \varepsilon^i h^{(i)}_{\mu\nu} + \varepsilon^j \mathscr{L}_{\zeta^{(j)}} \bar{g}_{\mu\nu} + \mathscr{O}\left(\varepsilon^{j+1}\right).$$

At each order:

$$\Delta_L h^{(i)}_{\ell\, \mu\nu} = S^{(i)}_{\ell\, \mu\nu}$$

- in polar sector: **seven** equations for **four** RW gauge invariant variables
- in axial sector: **three** equations for **two** RW gauge invariant variables

$$
\begin{aligned}
\Delta_L {}^{(i)}h_{\ell\, tt} &= \left[\frac{(2A+rA')^2-2(rA')^2+2(\ell-1)(\ell+2)A}{4r^2A} + \left(\frac{A'}{4}-\frac{A}{r}\right)\partial_r - \frac{A}{2}\partial_{rr}\right] {}^{(i)}f_{\ell\, tt} + \left[\frac{AA'}{2}A\partial_r - \partial_{tt}\right] {}^{(i)}f_{\ell\, +} \\
&\quad -A\left[\frac{(2A+rA')^2-4A}{4r^2} + \frac{AA'}{4}\partial_r + \frac{1}{2}\partial_{tt}\right] {}^{(i)}f_{\ell\, rr} + \left[\left(\frac{A'}{2}+\frac{2A}{r}\right)\partial_t + A\partial_{tr}\right] {}^{(i)}f_{\ell\, tr}\,, \\
\Delta_L {}^{(i)}h_{\ell\, rr} &= \left[\frac{4A(1-A)+(rA')^2}{4r^2A^3} - \frac{A'}{4A^2}\partial_r + \frac{1}{2A}\partial_{rr}\right] {}^{(i)}f_{\ell\, tt} - \left[\left(\frac{A'}{2A}+\frac{2}{r}\right)\partial_r + \partial_{rr}\right] {}^{(i)}f_{\ell\, +} \\
&\quad + \left[\frac{(2A+rA')^2+2A\left(2rA'+(\ell-1)(\ell+2)\right)}{4r^2A} + \left(\frac{A'}{4}+\frac{A}{r}\right)\partial_r + \frac{1}{2A}\partial_{tt}\right] {}^{(i)}f_{\ell\, rr} - \left(\frac{A'}{2A^2}\partial_t + \frac{1}{A}\partial_{tr}\right) {}^{(i)}f_{\ell\, tr}\,, \\
\Delta_L {}^{(i)}h_{\ell\, tr} &= \frac{A}{r}\partial_t {}^{(i)}f_{\ell\, 11} + \left[\left(\frac{A'}{2A}-\frac{1}{r}\right)\partial_t - \partial_{tr}\right] {}^{(i)}f_{\ell\, +} + \frac{\ell(\ell+1)}{2r^2} {}^{(i)}f_{\ell\, tr}\,, \\
\Delta_L {}^{(i)}h_{\ell\, +} &= \left[-\frac{2rA'+\ell(\ell+1)}{4A} + \frac{r}{2}\partial_r\right] {}^{(i)}f_{\ell\, tt} + \frac{A}{2}\left[\left(2A+3rA'+\frac{\ell(\ell+1)}{2}\right) + rA\partial_r\right] {}^{(i)}f_{\ell\, rr} \\
&\quad + \frac{1}{2}\left[(\ell-1)(\ell+2) - r(4A+rA')\partial_r - r^2A\partial_{rr} + \frac{r^2}{A}\partial_{tt}\right] {}^{(i)}f_{\ell\, +} - r\partial_t {}^{(i)}f_{\ell\, tr}\,, \\
\Delta_L {}^{(i)}h_{\ell\, t\theta} &= \frac{1}{2}\left[\left(A'+A\partial_r\right) {}^{(i)}f_{\ell\, tr} - A\partial_t {}^{(i)}f_{\ell\, rr} - \partial_t {}^{(i)}f_{\ell\, +}\right], \\
\Delta_L {}^{(i)}h_{\ell\, r\theta} &= \frac{2A+rA'}{4r} {}^{(i)}f_{\ell\, rr} - \frac{1}{2}\partial_r {}^{(i)}f_{\ell\, +} - \frac{1}{2A}\partial_t {}^{(i)}f_{\ell\, tr} + \frac{1}{2A}\left(-\frac{2A+rA'}{2rA} + \partial_r\right) {}^{(i)}f_{\ell\, tt}\,, \\
\Delta_L {}^{(i)}h_{\ell\, -} &= \frac{1}{4}\left(\frac{1}{A} {}^{(i)}f_{\ell\, tt} - A\, {}^{(i)}f_{\ell\, rr}\right).
\end{aligned}
$$

General approach to gravitational perturbations (2)

1. At each order there is only **one scalar gravitational degree of freedom** (for polar/axial perturbations, and for a given multipole ℓ) satisfying (in)homogeneous linear wave equation with a potential (to be determined)

$$\tilde{\Box}_\ell \Phi_\ell^{(i)}(t,r) := r\left(-\bar{\Box} + V_\ell\right)\frac{\Phi_\ell^{(i)}(t,r)}{r} = \tilde{S}_\ell^{(i)} \tag{1}$$

2. RW variables $f_{\ell\,+}^{(i)}, f_{\ell\,rr}^{(i)}, f_{\ell\,tr}^{(i)}, f_{\ell\,rr}^{(i)}$ are given as linear combinations of $\Phi_\ell^{(i)}$ and its derivatives (+ source functions at nonlinear orders):

$$f_{\ell\,+}^{(i)} = B\Phi_\ell^{(i)} + C\partial_t\Phi_\ell^{(i)} + D\partial_r\Phi_\ell^{(i)} + E\partial_{tr}\Phi_\ell^{(i)} + F\partial_{rr}\Phi_\ell^{(i)} + \alpha_\ell^{(i)}(t,r)\,, \tag{2}$$

$$f_{\ell\,rr}^{(i)} = \cdots + \beta_\ell^{(i)}(t,r)\,, \qquad f_{\ell\,tr}^{(i)} = \cdots + \gamma_\ell^{(i)}(t,r)$$

3. Satisfying (perturbative) Einstein equations fixes the potential V_ℓ and the coefficient functions in the equations above **uniquely (!)**
4. The relations (2) can be inverted for $\Phi_\ell^{(i)}$. There is a **unique (!)** way compatible with the ADM initial problem formulation. This also gives the source $\tilde{S}_\ell^{(i)}$ in (1) **uniquely (!)**

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$ (an easy way to the Zerilli equation)

master wave equation:

$$\tilde{\Box}_\ell \Phi_\ell^{(i)} := \frac{1}{A}\partial_{tt}\Phi_\ell^{(i)} - A\,\partial_{rr}\Phi_\ell^{(i)} - A'\,\partial_r\Phi_\ell^{(i)} + \left(\frac{A'}{r} + V_\ell\right)\Phi_\ell^{(i)} = \tilde{S}_\ell^{(i)}$$

potential (the celebrated Zerilli potential in the Schwarzschild case):

$$V_\ell = \frac{\ell(\ell+1)}{r^2} - \frac{A'}{r} + \underbrace{(2A - rA' - 2)}_{-6M/r}\frac{2A\,(rA'-2) - (rA')^2 + \ell^2(\ell+1)^2}{r^2\,(2A - rA' - \ell(\ell+1))^2}$$

and RW variables in terms of the master scalar variable (and source functions at nonlinear orders):

$$f_{\ell\,+}^{(i)} = A\partial_r\Phi_\ell^{(i)} + \frac{1}{r}\left(\frac{\ell(\ell+1)}{2} - \frac{2A - rA' - 2}{2A - rA' - \ell(\ell+1)}A\right)\partial_t\Phi_\ell^{(i)} + \alpha_\ell^{(i)}(t,r)$$

$$f_{\ell\,rr}^{(i)} = \cdots + \beta_\ell^{(i)}(t,r)$$

$$f_{\ell\,tr}^{(i)} = \cdots + \gamma_\ell^{(i)}(t,r)$$

Perturbations of spherically symmetric spaces, $A = 1 + \kappa r^2/\ell^2 - 2M/r$

$$\tilde{\Box}_\ell \Phi_\ell^{(i)} := \frac{1}{A}\partial_{tt}\Phi_\ell^{(i)} - A\,\partial_{rr}\Phi_\ell^{(i)} - A'\,\partial_r\Phi_\ell^{(i)} + \left(\frac{A'}{r} + V_\ell\right)\Phi_\ell^{(i)} = \tilde{S}_\ell^{(i)}$$

The master variable in terms of RW potentials - the **unique** form compatible with the ADM initial problem formulation:

$$\Phi_\ell^{(i)} = \frac{2r}{\ell(\ell+1)}\left(f_{\ell\,+}^{(i)} + 2A\,\frac{A f_{\ell\,rr}^{(i)} - r\partial_r f_{\ell\,+}^{(i)}}{\ell(\ell+1) - 2A + rA'}\right)$$

The sources $\tilde{S}_\ell^{(i)}$ source at higher orders can be read off accordingly

To fix the source functions $\alpha_\ell^{(i)}$, $\beta_\ell^{(i)}$ and $\gamma_\ell^{(i)}$ we write them down as linear combinations of the sources $S_{\ell\,\mu\nu}^{(i)}$ and their first derivatives. Fixing $3\times 7\times 3 = 63$ function coefficients of these linear combinations is a technical task. It turns out that 54 functions (out of 63) are fixed in terms of 9 free functions. Moreover, in the resulting expressions, coefficients of these 9 free functions are identically zero due to the identities for the sources, thus the final expressions are **uniquely** defined:

$$\alpha_\ell^{(i)} = -\frac{2A\left(r^2\left(A^{-1}S_{\ell\,tt}^{(i)} - AS_{\ell\,rr}^{(i)}\right) + 2S_{\ell\,+}^{(i)}\right)}{\ell(\ell+1)\left(\ell(\ell+1) - 2A + rA'\right)}$$

$$\beta_\ell^{(i)} = \frac{1}{A}\left(r\partial_r\alpha_\ell^{(i)} - \frac{\ell(\ell+1) - 2A + rA'}{2A}\alpha_\ell^{(i)}\right)$$

$$\gamma_\ell^{(i)} = \frac{r}{A}\partial_t\alpha_\ell^{(i)} + \frac{2r^2}{\ell(\ell+1)}S_{\ell\,tr}^{(i)}$$

Identities for the sources $S^{(i)}_{\mu\nu}$ - crucial for the consistency of higher orders of perturbation expansion

Taking the background divergence of perturbation Einstein equations

$$\bar{\nabla}\left(\Delta_L h^{(i)}_{\ell\,\mu\nu} - S^{(i)}_{\ell\,\mu\nu}\right) = 0$$

gives (three/one) identities in (polar/axial sectors) for the sources $S^{(i)}_{\mu\nu}$ (!)

Final conclusions

- **The hard part of perturbative Einstein equations (PDEs) can be reduced to only one scalar wave equation (for each polarization mode) and some linear algebra (!)**
- Crucial ingredients:
  - gauge invariance - implemented iteratively, thus Regge-Wheeler definitions of gauge invariants are sufficient
  - *ansatzes* for the form of solution (for RW gauge invariants and source functions (particular solutions of linear inhomogeneous system))
  - identities for the sources (inhomogeneous terms) in perturbative Einstein equations
- Although the scheme is conceptually simple its actual realization was rather unthinkable in pre- computer algebra era
- Now, it is time for practical applications (apart from AdS perturbation that motivated this approach) in the context of black holes stability, nonlinear gravitational waves and nonlinear quasinormal modes couplings, after introducing matter to the scheme also hopefully extreme-mass inspirals, accretion, cosmological perturbations, etc.