What Cauchy data evolve to stationary space-times?

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The Cauchy problem for the vacuum equations

The data for the Cauchy problem are

- a three manifold S with a Riemannian metric h_{ij}, which raises and lowers indices and has Levi-Civita covariant derivative D_i and Ricci scalar R, and
- a symmetric tensor K_{ij}, with trace $K = h^{ij} K_{ij}$,
- together satisfying the *vacuum constraints*:

$$D_j K_i{}^j - D_i K = 0 = R - K_{ij} K^{ij} + K^2.$$

These evolve to give a space-time M with Ricci-flat Lorentzian four-metric $g_{\alpha\beta}$ and Levi-Civita covariant derivative ∇_α containing a copy of S as a space-like surface with induced metric h_{ij} and second fundamental form K_{ij}.

The question we pose is: **what conditions on the data lead to a stationary space-time?**

Killing vectors

If M admits a Killing vector K^α then K_α satisfies

$$\nabla_\alpha K_\beta = M_{\alpha\beta} = -M_{\beta\alpha}$$

where also

$$\nabla_\alpha M_{\beta\gamma} = \mathcal{R}_{\beta\gamma\alpha\delta} K^\delta,$$

where $\mathcal{R}_{\beta\gamma\alpha\delta}$ is the four-dimensional Riemann tensor.

Call these the *The Killing equations* and note that they have the form of a linear system for the pair $(K_\alpha, M_{\alpha\beta})$.

The Killing Equations continued

To illustrate our method note that the Killing equations can be interpreted as covariant constancy of the section $\Phi_A := (K_\alpha, M_{\alpha\beta})^T$ of the rank ten vector bundle $\Lambda^1(M) \oplus \Lambda^2(M)$ in a particular linear connection:

$$\mathcal{D}_\alpha \Phi_B := \nabla_\alpha \Phi_B - \Gamma_{\alpha B}{}^C \Phi_C = 0,$$

for suitable $\Gamma_{\alpha B}{}^C$.

If there is a solution Φ_A then also

$$0 = \mathcal{D}_{[\alpha} D_{\beta]} \Phi_A =: \Omega_{\alpha\beta A}{}^B \Phi_B,$$

and candidates for Φ_A must lie in the kernel of the curvature Ω of this connection.

Introducing KIDs

Perform a $(3+1)$-split of the Killing equations: write n^α for the unit, future-pointing normal to S, then K^α defines on S a scalar field N and a vector field Y^i by

$$K^\alpha = Nn^\alpha + Y^i,$$

with a slight abuse of notation.

Beig and Chruściel (arXiv:gr-qc/9604040) call the pair (N, Y^i) a *KID* (standing for **K***illing* **I***nitial* **D***ata*, and the Killing equations impose equations on the KID.

The KID equations

Beig and Chruściel derive the equations

$$
\begin{aligned}
D_{(i}Y_{j)} &= -NK_{ij} \\
D_iD_jN + \mathcal{L}_Y K_{ij} &= N(R_{ij} + KK_{ij} - 2K_{im}K_j{}^m),
\end{aligned}
$$

from the Killing equations, where R_{ij} is the Ricci tensor of h_{ij} and $\mathcal{L}_Y$ is Lie-derivative along Y.

These are the *KID equations*. Beig, Chruściel and Schoen (arXiv:gr-qc/0403042) show that KIDs are not generic, that is that not all vacuum data admit KIDs, and give some necessary conditions. Our aim is to sharpen and interpret these conditions.

The idea: **use the Killing equations to prolong the KID equations to a linear system, and deduce conditions for existence of solutions.**

KIDs prolonged

We prolong the KID equations by introducing new variables $V_i := D_i N$ and $M_{ij} = D_{[i} Y_{j]}$ to obtain a linear first-order system:

$$
\begin{aligned}
D_i N &= V_i && (1)\\
D_i Y_j &= M_{ij} - N K_{ij} && (2)\\
D_i V_j &= N(R_{ij} + K K_{ij}) - Y^k D_k K_{ij} - K_{im} M_j{}^m - K_{jm} M_i{}^m && (3)\\
D_i M_{jk} &= -2 N D_{[j} K_{k]i} - R_{jki\ell} Y^\ell - 2 V_{[j} K_{k]i}. && (4)
\end{aligned}
$$

Here (1) follows from the definition of V_i, (2) follows from (1) and the definition of M_{ij}, (3) follows from (1), the definition of V_i and M_{ij} and expanding the Lie derivative, and (4) follows by commuting derivatives on Y_i. Here $R_{jki\ell}$ is the three-dimensional Riemann tensor.

This is the linear system that we wish to study.

The system

Introduce the ten component column-vector $\Psi_\alpha = (N, Y_j, V_j, M_{jk})^T$, which is a section of the vector bundle $\Lambda^0(S) \oplus \Lambda^1(S) \oplus \Lambda^1(S) \oplus \Lambda^2(S)$. Note that, given the Cauchy data, Ψ_α is equivalent to the KID and its first derivatives.
Now the system can be written as

$$\mathcal{D}_i \Psi_\alpha := D_i \Psi_\alpha - \Gamma_{i\alpha}{}^{\beta} \Psi_\beta = 0, \tag{5}$$

with $\Psi_\beta = (N, Y_p, V_p, M_{pq})^T$ and $\Gamma_{i\alpha}{}^{\beta}$ a matrix with tensor entries involving K_{ij}, $D_k K_{ij}$ and $R_{jki\ell}$.

This system is the condition for covariant-constancy of Ψ_α in a linear connection. Any solution must annihilate the curvature of this connection and this condition will be a set of linear equations on Ψ_α, that is on the KID and its first derivative.

To find the conditions we need only commute derivatives on the system (1)-(4). By construction, commutation on N and Y_i give identities. Commutation on V_i and M_{jk} give two sets of conditions:

Conditions on the KID and its first derivative

These conditions are the vanishing of the following two trace-free symmetric tensors:

$$C^{(1)}_{mn} :=$$

$$N\epsilon^{ij}{}_m(D_iR_{jn}+K_{,i}K_{jn}+KD_iK_{jn}+2K_i{}^pD_pK_{jn}+K_j{}^pD_nK_{pi}+K_n{}^pD_jK_{pi})$$

$$-\epsilon^{ij}{}_mY^pD_pD_iK_{jn}$$

$$+\epsilon^{ij}{}_m(V_i(R_{jn}+KK_{jn})-V_pK_j{}^pK_{ni}+K_n{}^pK_{pi}V_j)+\epsilon^{ij}{}_nG_{jm}V_i$$

$$-\epsilon^{ij}{}_m(M_j{}^pD_iK_{np}+M_n{}^pD_iK_{jp}+M_i{}^pD_pK_{jn})=0,$$

and

$$C^{(2)}_{mn} := N\epsilon^{ij}{}_m\epsilon^{pq}{}_n(-2D_iD_pK_{qj}-2K_{qj}(R_{ip}+KK_{ip}))-2N(G_{in}K^i{}_m-KG_{mn})$$

$$+2\epsilon^{ij}{}_m\epsilon^{pq}{}_nK_{qj}Y^kD_kK_{ip}-2Y^iD_iG_{mn}$$

$$-\epsilon^{ij}{}_m\epsilon^{pq}{}_n(2V_pD_iK_{qj}+2V_iD_pK_{qj})$$

$$+2\epsilon^{ij}{}_m\epsilon^{pq}{}_nK_{qj}(K_{is}M_p{}^s+K_{ps}M_i{}^s)+2G_{mi}M^i{}_n+2G_{ni}M^i{}_m=0.$$

Origin of the conditions

We've obtained ten linear expressions in the ten unknowns (N, Y_i, V_i, M_{ij}) with coefficients obtained from the Cauchy data, and there will be no solutions, and therefore no KIDs, if the relevant determinant, call it Δ, is nonzero.

Given a Killing vector K^α in a vacuum space-time, so that the Riemann tensor $R_{\alpha\beta\gamma\delta}$ equals the Weyl tensor $C_{\alpha\beta\gamma\delta}$, necessarily

$$0 = \mathcal{L}_K C_{\alpha\beta\gamma\delta} = K^\epsilon \nabla_\epsilon C_{\alpha\beta\gamma\delta} + C_{\alpha\beta\gamma\epsilon} M_\delta{}^\epsilon + C_{\alpha\beta\epsilon\delta} M_\gamma{}^\epsilon$$

$$+ C_{\alpha\epsilon\gamma\delta} M_\beta{}^\epsilon + C_{\epsilon\beta\gamma\delta} M_\alpha{}^\epsilon.$$

This is a set of ten linear equations in the components of $(K_\alpha, M_{\alpha\beta})$ i.e. in the KID and its first derivatives. **Claim:** perform a $(3+1)$-decomposition of this set to conclude that these are the same ten conditions previously found.

Can we *interpret* $\Delta = 0$, the vanishing of the determinant this way too?

Interpreting the vanishing of the determinant

It's well-known that in dimension four there are four real scalar invariants of the Weyl tensor which are the real and imaginary parts of the complex scalar invariants of the Weyl spinor

$$I := \psi_{ABCD}\psi^{ABCD}, \quad J := \psi_{AB}{}^{PQ}\psi_{PQCD}\psi^{ABCD},$$

and it's clear that any Killing vector K^α satisfies

$$\mathcal{L}_K I = 0 = \mathcal{L}_K J.$$

Thus K^α annihilates the gradient of four real scalars and there can be no Killing vector if the four scalars are functionally independent, that is if

$$\tilde{\Delta} := dI \wedge dJ \wedge d\bar{I} \wedge d\bar{J} \neq 0.$$

Claim: This is the same condition as $\Delta \neq 0$.

This is now just a matter of checking.

A loose end

For an algebraically-special space-time one necessarily has a relation between the complex invariants, namely

$$I^3 - 6J^2 = 0,$$

and so $\tilde{\Delta} = 0$ and $\Delta = 0$.

Now there will exist candidate KIDs, annihilating $dI \wedge d\bar{I}$. It is an outstanding problem to decide under what conditions these are in fact KIDs.

It remains a possibility that all weakly-asymptotically-simple, algebraically-special space-times are in fact stationary.