On some recent results on total masses in GR with $\Lambda > 0$

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1. Introduction: Total energy-momenta
   1.1 Generalities
   1.2 Asymptotically flat spacetimes
   1.3 Asymptotically de Sitter spacetimes
2. Total mass of closed universes with $\Lambda \geq 0$ (LBSz, CQG '12, '13),
3. A Bondi type mass in asymptotically de Sitter spacetimes (LBSz, P. Tod, CQG '15)
4. Summary

1. INTRODUCTION: Total energy-momenta in GR

1.1 Generalities

No well defined energy-momentum density for gravity
Total mass/energy-momentum associated with the 'whole system'
Why? – energy-momentum/mass basic notions in physics; useful tools in geometric analysis (of spacetimes)
Open questions: What is

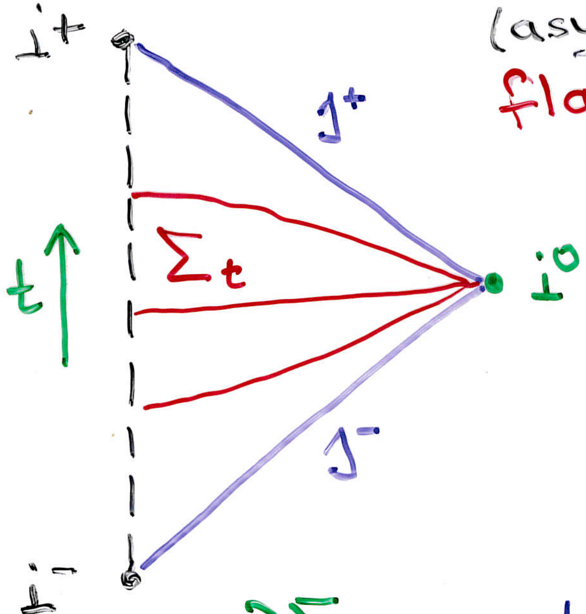
- the domain ('system') ($\mathcal{S}$, Σ – how to choose these?);
- the specific expression to use (which 'superpotential' or 'energy-momentum complex' to choose?);
- the generator for energy-momentum (what to mean by 'translation'?)

1.2 Motivation: Asymptotically flat spacetimes ($\Lambda = 0$)

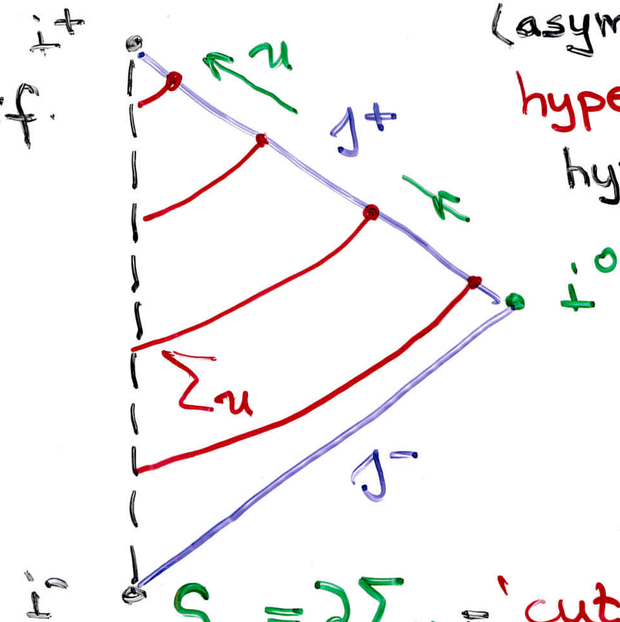
Significant contributions from the Polish relativity community:
Trautman; Bizon, Chrusciel, Jezierski, Kijowski, Malec, Tafel, ...

- The domain:

- **The domain:** foliation by



(asymptotically) flat hypersurf. Σ_t



(asymptotically) hyperboloidal hypersurfaces Σ_u

$S_\infty = \partial \Sigma_t$ – at i^0

$S_\infty = \partial \Sigma_u$ – 'cut' of $\mathcal{J}^+$

Yields: ADM – 4 momentum

BS – 4 momentum

• The superpotential: Nester–Witten 2-form (Horowitz–Tod)

$$u(\lambda,\bar\mu)_{ab}:=\frac{\mathrm{i}}{2}(\bar\mu_{A'}\nabla_{BB'}\lambda_A-\bar\mu_{B'}\nabla_{AA'}\lambda_B)$$

– essentially 'real', for which

$$\nabla_{[a}u(\lambda,\bar\mu)_{bc]}=S(\lambda,\bar\mu)_{abc}-\frac{1}{2}\lambda^D\bar\mu^{D'}G_d{}^e\frac{1}{3!}\varepsilon_{eabc}$$

– the 'superpotential equation', where

$S(\lambda,\bar\mu)_{abc}$ – homogeneous quadratic in the 1st derivatives ('Sparling's 3-form'),

G_{ab} – Einstein's tensor.

• The generator: λ^A, μ^A – spinor constituents of the ADM/BMS translations at infinity

These + Einstein's equations yield:

$$H[\lambda,\bar\lambda]:=\frac{2}{\kappa}\oint_{\partial\Sigma}u(\lambda,\bar\lambda)_{ab}=$$
$$=\int_\Sigma\Bigl(\frac{4}{\kappa}t^{AA'}t^{BB'}t^{CC'}(\mathcal{D}_{(AB}\lambda_{C)})(\mathcal{D}_{(A'B'}\bar\lambda_{C')})+\lambda^A\bar\lambda^{A'}T_{ab}t^b-$$
$$-\frac{8}{3\kappa}t^{BB'}(\mathcal{D}_{B'A}\lambda^A)(\mathcal{D}_{BA'}\bar\lambda^{A'})\Bigr)\mathrm{d}\Sigma$$

Witten's gauge condition $\mathcal{D}_{A'A}\lambda^A=0$,

Existence and uniqueness of the solution with the translations as boundary condition both on AF and AH Σ.

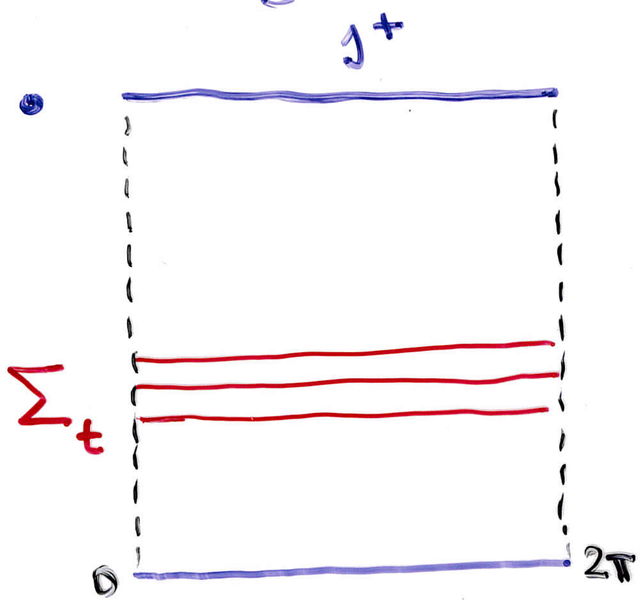
Thus: dominant energy condition + Witten's gauge condition yield positivity and rigidity of the functional H; and of the ADM/BS 4-momenta

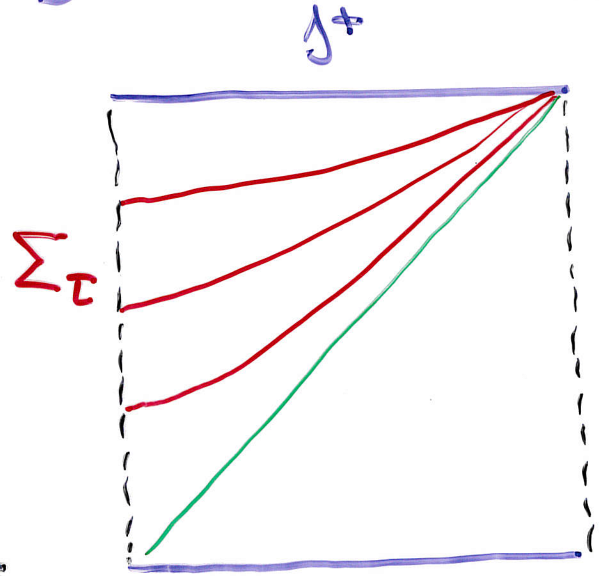
$$\mathtt{P}_{\underline{A}\,\underline{A}'}\lambda^{\underline{A}}_\infty\bar\lambda^{\underline{A}'}_\infty:=H[\lambda,\bar\lambda]$$

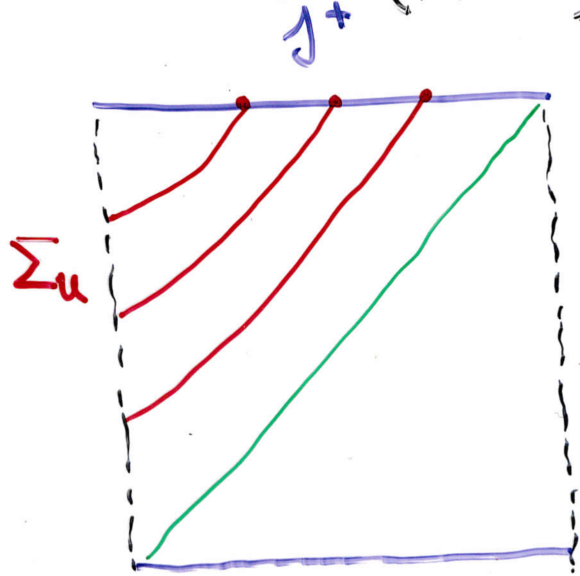
and masses

$$\mathtt{m}^2:=\mathtt{P}_{\underline{A}\,\underline{A}'}\mathtt{P}_{\underline{B}\,\underline{B}'}\epsilon^{\underline{A}\,\underline{B}}\epsilon^{\underline{A}'\underline{B}'}.$$

1.3 Asymptotically de Sitter spacetimes ($\Lambda > 0$)







- global foliation by Cauchy surf.

$\Sigma_t \simeq S^3$

(total mass for closed Σ with $\Lambda \geq 0$
– L.B.Sz 2012, 2013)

↑ Sec. 2

partial Cauchy surf.

$\Sigma_\tau \simeq \mathbb{R}^3$

- asympt. intrinsic. flat

(Abbot - Deser 1982
Kastor - Traschen 2002)

partial Cauchy

$\Sigma_u \simeq \mathbb{R}^3$

- asympt. hyperboloidal

(LBSz - P. Tod 2015
Chrusciel - Ifsits 2016
Saw 2016)

↑ Sec. 3

2. TOTAL MASS OF CLOSED UNIVERSES WITH $\Lambda \geq 0$

• ADM/BS: We saw

$$\mathrm{P}_{\underline{A}\,\underline{A}'}\lambda^{\underline{A}}_{\infty}\bar\lambda^{\underline{A}'}_{\infty}=\frac{\sqrt{2}}{\kappa}\Vert\mathcal{D}_{(AB}\lambda_{C)}\Vert^2_{L_2}+\int_\Sigma t^aT_{ab}\lambda^B\bar\lambda^{B'}\mathrm{d}\Sigma$$

in Witten's gauge

• Closed universe:

Domain: Σ – compact Cauchy surface in glob. hyperbolic (M, g_{ab});

NO 'superpotential' and natural 'translations'.

But:

$$\mathtt{M}:=\inf_\lambda\Big\{\frac{\sqrt{2}}{\kappa}\Vert\mathcal{D}_{(AB}\lambda_{C)}\Vert^2_{L_2}+\int_\Sigma t^a\big(T_{ab}+\frac{\Lambda}{\kappa}g_{ab}\big)\lambda^B\bar\lambda^{B'}\mathrm{d}\Sigma\Big\}\geq 0$$

$\lambda^A\in C^\infty(\Sigma,\mathbb{S}^A)$, normalization: $\Vert\lambda^A\Vert^2_{L_2}=\sqrt{2}$.

N.B.:

1. In ADM/BS: $\mathtt{m}\geq\mathtt{M}$, – lower bound for the total masses;
2. $\kappa\mathtt{M}\geq\Lambda\geq 0$, and equality in locally de Sitter;
3. $8\Vert\mathcal{D}_{A'A}\lambda^A\Vert^2_{L_2}\geq 3\kappa\,\mathtt{M}\,\Vert\lambda^A\Vert^2_{L_2}$.

– is there a smooth minimizer spinor field?

Proposition: $\exists\lambda^A\in C^\infty(\Sigma,\mathbb{S}^A)$ such that

- $2\|\mathcal{D}_{A'A''}\lambda^A\|^2_{L_2}=\frac{3\kappa}{4}\mathtt{M}\|\lambda^A\|^2_{L_s}$ ('minimizer spinor field'),
- λ^A is the 1st eigenspinor of $\mathcal{D}^*\mathcal{D}$: $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}\lambda^B=\frac{3}{4}\kappa\mathtt{M}\lambda^A$.

Hence:

- $\mathtt{M}$ is just the 1st eigenvalue of $\mathcal{D}^*\mathcal{D}$
- Witten's eq. admits a solution iff $\mathtt{M}=0$, thus no solution for $\Lambda>0$.

• The rigidity properties of $\mathtt{M}$:

Theorem: Let the matter fields satisfy the dominant energy cond.

- If $\Lambda=0$, then $\mathtt{M}=0$ iff (M,g_{ab}) is flat; and $\Sigma\approx S^1\times S^1\times S^1$.
- If $\Lambda>0$, then $\kappa\mathtt{M}=\Lambda$ iff (M,g_{ab}) is loc. isometric with de Sitter; and $\Sigma\approx S^3/G$, where $G\subset SU(2)\approx S^3$ discrete.

– analogous to the rigidity part of the positive energy theorems, thus

i. $\mathtt{M}$ is a positive definite measure of the strength of the grav. field;

ii. the physical dimension of $\mathtt{M}\mathrm{vol}(\Sigma)$ is mass;

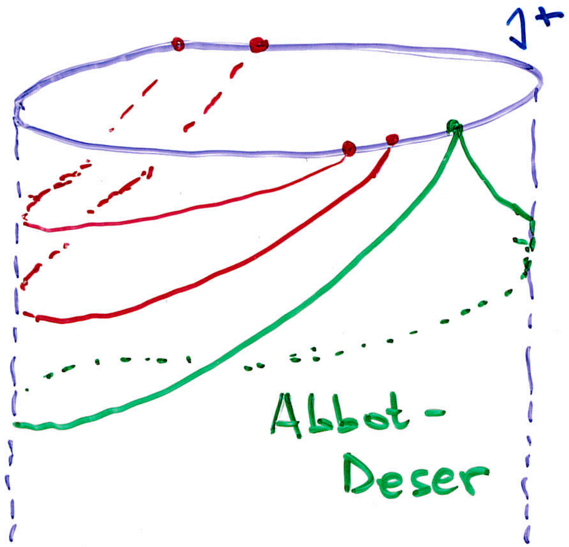
iii. given by the formula for the pos. lower bound for the ADM/BS mass.

Thus, natural interpretation: Total mass (density) of the closed universes.

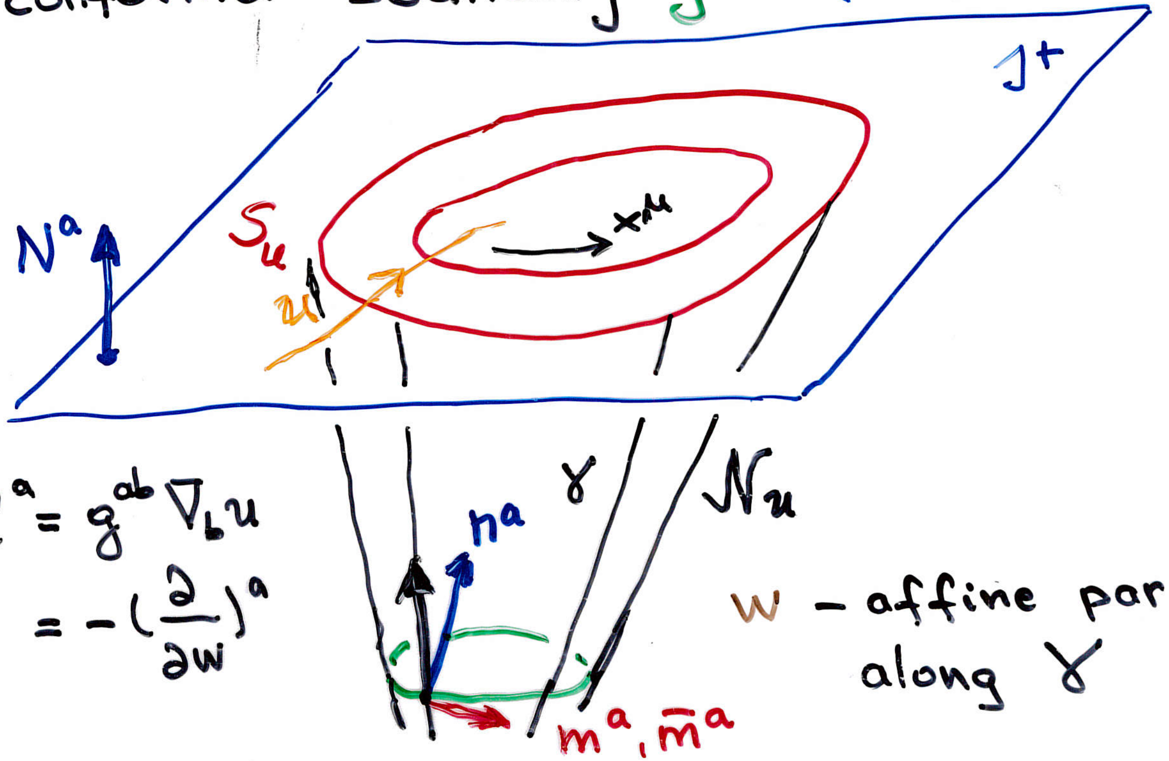
4. A BONDI TYPE MASS IN ASYMPT. DE SITTER

$\Lambda > 0$, $\exists$ C^∞ conformal boundary $\mathcal{J}^+$ (with P. Tod)

- The domain:



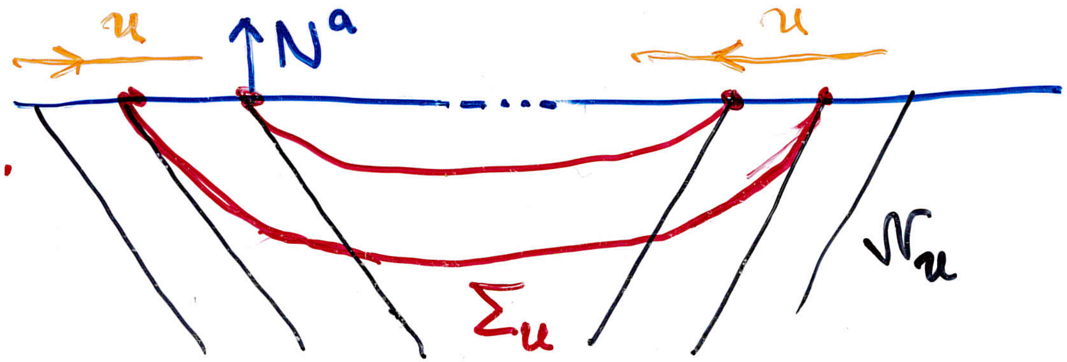
de Sitter



$$l^a = g^{ab}\nabla_b u = -\left(\frac{\partial}{\partial w}\right)^a$$

w - affine par. along γ

Induced data set on Σ: asymptotically hyperb.



- The ‘superpotential’: The Nester–Witten form, hence

$$H[\lambda,\bar\lambda]=\int_\Sigma\Bigl(\frac{4}{\kappa}t^{AA'}t^{BB'}t^{CC'}(\mathcal{D}_{(AB}\lambda_{C)})(\mathcal{D}_{(A'B'}\bar\lambda_{C')})+ \\ +\lambda^A\bar\lambda^{A'}(T_{ab}+\frac{\Lambda}{\kappa}g_{ab})t^b-\frac{8}{3\kappa}t^{BB'}(\mathcal{D}_{B'A}\lambda^A)(\mathcal{D}_{BA'}\bar\lambda^{A'})\Bigr)\mathrm{d}\Sigma$$

– The Λ term must be subtracted!

Gibbons etal: ‘Renormalized’ Witten equations for $(\alpha^A,\bar\beta^{A'})$:

$$\mathcal{D}_{A'A}\alpha^A+\frac{3}{2}\mathrm{i}\sqrt{\frac{\Lambda}{6}}\bar\beta_{A'}=0,\qquad \mathcal{D}_{AA'}\bar\beta^{A'}+\frac{3}{2}\mathrm{i}\sqrt{\frac{\Lambda}{6}}\alpha_A=0$$

Hence:

$$H[\alpha,\bar\alpha]+H[\beta,\bar\beta]=\int_\Sigma\Bigl(t^aT_{ab}(\alpha^B\bar\alpha^{B'}+\beta^B\bar\beta^{B'})+ \\ +\frac{4}{\kappa}t^{AA'}t^{BB'}t^{CC'}(\mathcal{D}_{(AB}\alpha_{C)}\mathcal{D}_{(A'B'}\bar\alpha_{C')}+\mathcal{D}_{(AB}\beta_{C)}\mathcal{D}_{(A'B'}\bar\beta_{C')})\Bigr)\mathrm{d}\Sigma.$$

Questions:

- ∘ boundary conditions for $(\alpha^A,\bar\beta^{A'})$?
- ∘ existence/uniqueness of the solutions of the Witten equations?
- ∘ meaning of the integral?

- Great surprise: NO freely specifiable generators!

The Witten equation + finiteness determine the boundary conditions:

- $\mathrm{i}\bar\beta_{A'} = \sqrt{2}N_{A'A}\alpha^A$ – links the two spinors on the cut of $\mathscr{I}^+$;
- $\eth\alpha_0 + \sigma\alpha_1 = 0, \quad \eth'\alpha_1 - \sigma\alpha_0 = 0$ – 2-surface twistor equations.

Proposition: With such a boundary condition the renormalized Witten equation admits a unique solution.

– technical issue: the traditional weighted Sobolev spaces are NOT the appropriate function spaces.

Theorem: Let the dominant energy condition holds. Then

- $H^*: \mathbb{T}(\mathcal{S}) \to \mathbb{C}$ is a finite, non-negative Hermitian bilinear form,
- it is zero iff $D(\Sigma)$ is locally de Sitter,
- the symmetry group of $\mathbb{T}(\mathcal{S})$ is $SL(2,\mathbb{H}) \times \mathbb{R}^+ \approx SO(1,5) \times \mathbb{R}^+$,
- the structure of H^* in a natural basis of $\mathbb{T}(\mathcal{S})$ is

$$H^* = \begin{pmatrix} \mathbb{P} & \mathbb{Q} \\ -\bar{\mathbb{Q}} & \bar{\mathbb{P}} \end{pmatrix},$$

- with a natural volume form $\varepsilon_{\alpha\beta\gamma\delta}$ on $\mathbb{T}(\mathcal{S})$ the mass is $\mathtt{m}^4 := \det(H^*)$ $= (\det(\mathbb{P}) - \mathbb{Q}\bar{\mathbb{Q}})^2$: Then $\mathtt{m} = 0$ iff $D(\Sigma)$ is locally de Sitter.

5. SUMMARY

- The quantity

$$\mathtt{M} := \inf\Big\{\frac{\sqrt{2}}{\kappa}\|\mathcal{D}_{(AB}\lambda_{C)}\|^2_{L_2} + \int_\Sigma t^a T_{ab}\lambda^B\bar\lambda^{B'}\mathrm{d}\Sigma\Big\},$$

defined on the set of smooth spinor fields satisfying appropriate boundary and normalization conditions on AF/AH hypersurfaces in asymptotically flat spacetimes, provides a positive lower bound for $\mathtt{m}_{ADM}$, $\mathtt{m}_{BS}$.

- On closed Σ for the same $\mathtt{M}$ (with $\tilde T_{ab} := T_{ab} + \frac{\Lambda}{\kappa}g_{ab}$, $\Lambda \geq 0$):
  - $\mathtt{M}$ gives the first eigenvalue α_1^2 of the operator $2\mathcal{D}^{AA'}\mathcal{D}_{A'B}$:

$$\alpha_1^2 = \frac{3}{4}\kappa\,\mathtt{M},$$

  - Witten's gauge, $\mathcal{D}_{A'A}\lambda^A = 0$, admits a non-trivial sol. iff $\mathtt{M} = 0$.
  - For $\Lambda = 0$: $\mathtt{M} = 0$ iff (M, g_{ab}) is flat with toroidal Σ,
  - For $\Lambda > 0$: $\kappa\mathtt{M}$ is bounded from below by Λ, and $\mathtt{M} = \Lambda/\kappa$ iff (M, g_{ab}) is locally isometric with the de Sitter spacetime.

• On asymptotically hyperboloidal Σ (with $\Lambda > 0$): The Nester–Witten form yields a Hermitian bilinear functional $H^* : \mathbb{T}(\mathcal{S}) \to \mathbb{C}$, which is

- finite and non-negative if the dominant energy cond. holds,
- zero iff $D(\Sigma)$ is locally de Sitter,
- the mass,

$$\mathtt{m}^4 := \det(H^*) = \det \begin{pmatrix} \mathbb{P} & \mathbb{Q} \\ -\bar{\mathbb{Q}} & \bar{\mathbb{P}} \end{pmatrix} \geq 0,$$

  is zero iff $D(\Sigma)$ is locally de Sitter,
- the symmetry group of $\mathbb{T}(\mathcal{S})$ is $SL(2,\mathbb{H}) \times \mathbb{R}^+ \approx SO(1,5) \times \mathbb{R}^+$,

Open questions:

◦ Is there a natural reduction of $SL(2,\mathbb{H}) \times \mathbb{R}^+$ to $SL(2,\mathbb{C})$, or at least to $SL(2,\mathbb{H})$, to get 4-momentum or at least an invariant mass? How the de Sitter group emerges?

◦ The meaning of $\mathbb{Q}$?

◦ Is there an analog of Bondi's mass-loss?

◦ Could the 2-surface twistors be interpreted as 'spinor constituents' of some 'asymptotic (conformal) symmetries'? ...