REMARKS (mainly historical) ON THE THEORY OF CLASSICAL RELATIVISTIC SPINNING PARTICLES

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Plan

1. Foreword
2. Myron Mathisson's *Neue Mechanik materieller Systeme*
   (Acta Physica Polonica 6, 163-200 (1937))
3. Jan Weyssenhoff and Antoni Raabe
4. Theory of classical spinning particles in Cracow after 1947
5. Summary

Foreword

- Theory of classical spinning particles is one of evergreens in theoretical physics
- Classical point-like object with certain internal degrees of freedom related to angular momentum; its kinematics and dynamics
- It is a purely theoretical construct, like the classical point particle itself (very useful though), $\rightarrow$ some arbitrariness
- Large number of interesting models of this kind: vector, tensor, spinor, Grassmann, ... internal d.o.f. – see a review by A. Frydryszak[1], or ask Professor J. Lukierski
- Axiomatic Lagrangian definition, or a limiting case of another theory. In the latter case, we usually obtain eqs. of motion first, Lagrangian is secondary, if there is one at all
- The paper by M. Mathisson is an exception: no Lagrangian, and no limit. Instead, a dimensional transfiguration

[1] In *From Field Theory to Quantum Groups*, B. Jancewicz, J. Sobczyk (Eds.). World Scientific, 1996

Myron Mathisson (1897-1940)

(from Acta Physica Polonica B)

Born and educated in Warsaw

1930: PhD (Cz. Białobrzeski, UW)

1932: Habilitation (UW)

1936/37: at the Kazan University

1937/1939: in Cracow

1939-1940: in Cambridge (England)

12 published papers; the last one posthumously in 1941

"This work was found in an unfinished state among the papers left by Dr Myron Mathisson, who died on 13 September 1940. I have edited it and have added a summary. .... [P.A.M. Dirac.]"

Myron Mathisson

More information in:

1. Acta Phys. Polon. B. Proceedings Supplement, 1, 1-228 (2008)

Contains 17 articles presented at the International Conference Devoted to Myron Mathisson: His Life, Work, and Influence on Current Research (Warsaw, 2007). In particular:
T. Sauer and A. Trautman, *Myron Mathisson: What Little We Know of His Life*,
W. G. Dixon, *Mathisson's New Mechanics: Its Aims and Realisation*

2. Books and articles on the history of theoretical physics in Cracow by Bronisław Średniawa (1917-2014), student and collaborator of Jan Weyssenhoff, and later professor of theoretical physics at the Jagiellonian University

* * *

The original paper of 1937 has been republished as the Golden Oldie:
New Mechanics of Material Systems,
Gen. Relativ. Gravit. 42, 1011-1048 (2010) (in English)

New Mechanics of Material Systems

Minkowski space-time M

Timelike line L in M: $X^\alpha(s),\ \dot{X}^2 = 1,\ \dot{} \equiv d/ds$

Local coordinates $(s, \vec{z})$ in M around L: $x^\mu = X^\alpha(s) + z^\mu,\ \dot{X}z = 0$

An isolated piece of matter represented by smooth energy-momentum tensor $T^{\alpha\beta},\ \partial_\beta T^{\alpha\beta} = 0$. $T^{\alpha\beta}$ vanishes outside certain tube of finite width surrounding L.

Smooth functions $p_{\alpha\beta}(x)$ on M

$$\int d^4x\, T^{\alpha\beta}(x)\, p_{\alpha\beta}(x) = \int ds\, d^3z\, \sqrt{g}\, T^{'\alpha\beta}(s,\vec{z})\, p^{'}_{\alpha\beta}(s,\vec{z}),$$

$$p^{'}_{\alpha\beta}(s,\vec{z}) = p^{'}_{\alpha\beta}(s,0) + z^i\, \partial_i p^{'}_{\alpha\beta}(s,0) + \frac{1}{2}\, z^i z^k\, \partial^2_{ik} p^{'}_{\alpha\beta}(s,0) \ldots,$$

$$\int d^4x\, T^{\alpha\beta}(x)\, p_{\alpha\beta}(x) = \int ds\, \Big[\tilde{m}^{'\alpha\beta}(s) p^{'}_{\alpha\beta}(s,0) + \tilde{m}^{'\alpha\beta i}(s)\partial_i p^{'}_{\alpha\beta}(s,0) + \tilde{m}^{'\alpha\beta ik}(s)\partial^2_{ik} p^{'}_{\alpha\beta}(s,0) + \ldots\Big]$$

New Mechanics of Material Systems

$\partial_i|_{\vec{z}=0} \to \nabla_i|_{\vec{z}=0}$ by adding and subtracting appropriate terms

Returning to the Cartesian coordinates x^μ

$$\int_M d^4x\, T^{\alpha\beta}(x)\, p_{\alpha\beta}(x) = \int_L ds\, \big[m^{\alpha\beta}(s)\, p_{\alpha\beta}(X(s)) + m^{\alpha\beta\mu}(s)\, \partial_\mu p_{\alpha\beta}(X(s)) + m^{\alpha\beta\mu\nu}(s)\, \partial_\mu \partial_\nu p_{\alpha\beta}(X(s)) + \ldots \big] \qquad (\star)$$

The moments $m^{\alpha\beta\mu\nu\cdots}(s)$ are symmetric in $\alpha\,\beta$ as well as in $\mu\,\nu\ldots$, and orthogonal to $\dot{X}(s)$ in $\mu, \nu, \ldots$, i.e., $m^{\alpha\beta\mu\nu\cdots}(s)\,\dot{X}_\mu(s) = 0$, etc.

Weak equivalence:

$$T^{\alpha\beta}(x) \text{ on } M \quad \Longleftrightarrow \quad \{ m^{\alpha\beta}(s),\, m^{\alpha\beta\mu}(s),\, m^{\alpha\beta\mu\nu}(s), \ldots \} \text{ on } L$$

↗ the skeleton of the matter

This equivalence can be called the dimensional transfiguration

New Mechanics of Material Systems

Take $p_{\alpha\beta}(x) = \partial_\beta q_\alpha(x)$. Integrate by parts and use $\partial_\beta T^{\alpha\beta} = 0$ on the l.h.s. of formula ($\star$). The constraint on the skeleton ensues:

$$0 = \int_L ds\ \big[m^{\alpha\beta}(s)\,\partial_\beta q_\alpha(X(s)) + m^{\alpha\beta\mu}(s)\,\partial_\mu\partial_\beta q_\alpha(X(s)) + m^{\alpha\beta\mu\nu}(s)\,\partial_\mu\partial_\nu\partial_\beta q_\alpha(X(s)) + \ldots\big]$$

for arbitrary smooth functions $q_\alpha(x)$. Mathisson called it 'the variational equation of relativistic dynamics' (today $q_\alpha \to \delta q_\alpha$)

The pole-dipole case:

$$0 = \int_L ds\ \big[m^{\alpha\beta}(s)\,\partial_\beta q_\alpha(X(s)) + m^{\alpha\beta\mu}(s)\,\partial_\mu\partial_\beta q_\alpha(X(s))\big]$$

Use the decompositions

$$m^{\alpha\beta} = m\,\dot X^\alpha \dot X^\beta + M^\alpha \dot X^\beta + M^\beta \dot X^\alpha + {}^*m^{\alpha\beta},$$

$$m^{\alpha\beta\mu} = n^\mu\,\dot X^\alpha \dot X^\beta + n^{\alpha\mu} \dot X^\beta + n^{\beta\mu} \dot X^\alpha + {}^*m^{\alpha\beta\mu},$$

where all $M, n, {}^*m$ are orthogonal to $\dot X$ in all indices

New Mechanics of Material Systems

Mathisson assumed that $n^{\mu} = 0$. Next, he showed that

$$ {}^*m^{\alpha\mu} = 0,\ {}^*m^{\alpha\beta\mu} = 0,\ n^{\alpha\mu} = -n^{\mu\alpha},\ M^{\alpha} = \dot{n}^{\alpha\mu}\dot{X}_{\mu},\ \dot{m} = 0 $$

(well, I would not say the proof is straightforward). Finally,

$$ \underline{m\ddot{X}^{\mu} = 2n^{\mu\nu}\dddot{X}_{\nu}, \quad \dot{n}^{\mu\nu} = \left(\dot{X}^{\mu}n^{\nu\alpha} - \dot{X}^{\nu}n^{\mu\alpha}\right)\ddot{X}_{\alpha}} $$

↖ the Mathisson equations in the pole-dipole case

The proper time derivatives appear because $\dot{X}^{\mu}\partial_{\mu} = d/ds$

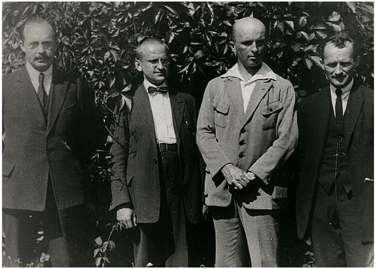
- In the modern notation $2n^{\mu\nu} \to S^{\mu\nu}$ (the spin tensor)
- Note the derivative of the third order
- Mathisson also considered: the quadrupole term, coupling to external gravitational and electromagnetic fields, and physical interpretation of the first two moments
- The approach was significantly improved in a series of papers by W. G. Dixon around 1970

Jan Weyssenhoff (1889-1972)

Cracow, 1911

(by courtesy of Dr. Hab. Piotr Dobosz)

Vilnius, 1926



(from the Library of Nicolaus Copernicus University, Toruń)

Born in Warsaw, educated in Cracow
1914-1919: at the University of Zurich
1916: PhD *Applications of quantum theory to rotating objects and the theory of paramagnetism*
1921: Habilitation in Cracow
1921-1935: at the University of Vilnius
since 1935: at the Jagiellonian U. in Cracow

Several textbooks, including *The Art of Playing Soccer* (Lviv, 1926. 408pp, 78 figures) recommended for schools

During the war, secret seminar on relativistic spin particles (A. Bielecki, A. Raabe, B. Średniawa). Lecturer at the Underground University

Antoni Raabe (1915-1942)

Born and educated in Warsaw

1938: MSc in physics (UW)

1938-39: volunteer at Department of Theoretical Physics, Jagiellonian University, Cracow. Worked with Mathisson and Weyssenhoff

1940-41: assistant at the University of Lviv. Moved there with J. Weyssenhoff

1941-42: back in Cracow with J. Weyssenhoff

In summer 1942 captured by Germans in a man-hunt in Cracow. Died in Auschwitz on September 7th, 1942

Coauthor of 3 papers (with J. Weyssenhoff): Nature (1946) – 1, Acta Physica Polonica (1947) – 2, including the renowned *Relativistic Dynamics of Spin-Fluids and Spin-Particles* (Acta Physica Polonica 9, 7-18 (1947))

Relativistic Dynamics of Spin-Fluids and Spin-Particles

Relativistic fluid in Minkowski space-time

$$T^{\mu\nu}(\vec{x},t) = p^{\mu}(\vec{x},t)\, u^{\nu}(\vec{x},t), \quad u^{\nu}u_{\nu} = 1$$

(energy-momentum tensor does not have to be symmetric!)
Four conserved currents:

$$\partial_{\nu}T^{\mu\nu} = 0, \quad \text{i.e.,} \quad \partial_0(p^{\mu}u^0) + \partial_i(p^{\mu}u^i) = 0$$

Abbreviations: $D_u f(\vec{x},t) \equiv \partial_{\mu}(fu^{\mu}), \quad d_u f(\vec{x},t) \equiv u^{\mu}\partial_{\mu} f$.
Thus $D_u p^{\mu} = 0$; $D_u f = 0$ means that the current fu is conserved.
(In the paper there is D_{τ}, d_{τ}, I have changed the notation for pedagogical reasons.)

Consider the currents of components of angular momentum, that is take for f the six functions $x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$. These currents are not conserved, $D_u(x^{\mu}p^{\nu} - x^{\nu}p^{\mu}) = u^{\mu}p^{\nu} - u^{\nu}p^{\mu}$ (unless we assume that $p||u$). This motivates us to endow the fluid with continuous 'spin charge' density $s^{\mu\nu}(\vec{x},t)$ such that

$$D_u s^{\mu\nu} = u^{\nu}p^{\mu} - u^{\mu}p^{\nu}, \quad s^{\mu\nu} = -s^{\nu\mu}, \quad s^{\mu\nu}u_{\nu} = 0$$

Relativistic Dynamics of Spin-Fluids and Spin-Particles

In consequence, $p^\mu = mu^\mu + d_u u_\nu \ s^{\nu\mu}$, where $m(\vec{x}, t) \equiv p^\nu u_\nu$.
$m(\vec{x}, t)$ is conserved: $D_u m = 0$

Now comes a step which can be tricky for those who are not familiar with hydrodynamics (including me): the derivation of evolution equations for the charges and for the velocity of a finite drop that travels with the fluid. This amounts to a kind of integration. Define

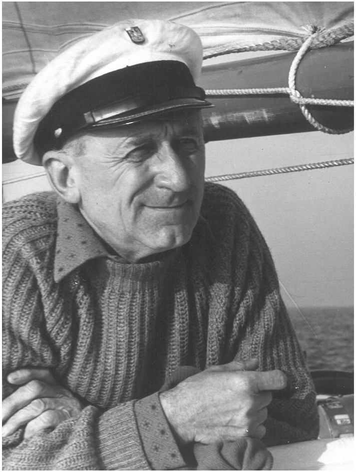
$$P^\mu = \int dV \ p^\mu, \quad S^{\mu\nu} = \int dV \ s^{\mu\nu}, \quad M = \int dV \ m,$$

and assume that the drop is so small that the velocity u can be regarded as constant in it. It turns out that $\int dV \ D_u s^{\mu\nu} = dS^{\mu\nu}/ds$, etc., where s is the proper time along the trajectory of the drop. The evolution equations have the form

$$M\dot{u}^\mu = \ddot{u}_\nu S^{\mu\nu}, \quad \dot{S}^{\mu\nu} = (S^{\nu\rho}u^\mu - S^{\mu\rho}u^\nu)\dot{u}_\rho.$$

$\dot{M} = 0$, and the momentum of the drop is $P^\mu = Mu^\mu + \dot{u}_\nu S^{\nu\mu}$.

Classical spinning particles in Cracow after 1947



(by courtesy of Dr. B. Średniawa)

Until 1964: J. Weyssenhoff, B. Średniawa, Z. Borelowski

solving Mathisson's equations; classical point particles with various internal d. o. f.

1979-1988: H. Arodź, K. Golec-Biernat

classical point particle with spin and color; Ehrenfest type classical limit for the Dirac equation with Yang-Mills or electromagnetic field

2008-2012: A. Staruszkiewicz, Ł. Bratek

'the fundamental relativistic rotator'

Classical spinning particles in Cracow after 1947

H. A., Th. W. Ruijgrok, *On Classical Limit of the Dirac Equation with an External Electromagnetic Field. Part I. Proper-Time Evolution, Lorentz Covariant Expectation Values and Classical Equations of Motion*, Acta Phys. Polon. B 19, 99-140 (1988)

Problem with Lorentz covariance in Ehrenfest's approach to classical limit: in general, the standard expectation values do not have clear transformation law because d^3x is not a scalar

1. Reformulation of the Dirac quantum mechanics

$$\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + ieA_\mu \right) \psi + im\,\psi = 0 \quad \rightarrow \quad \Gamma^\mu \left(\frac{\partial}{\partial s^\mu} + ieB_\alpha \right) \psi + im\,\psi = 0$$

in the coordinates $(s^\alpha) = (s, \vec{z})$. We take s for the evolution parameter. It is Lorentz invariant

Lorentz invariant scalar product and covariant expectation values

$$(\psi|\chi) = \int d^3z \sqrt{g}\ \overline{\psi}\Gamma^0\chi, \quad X^\mu(s) = (\psi|\hat{x}^\mu\psi), \text{ etc.}$$

Classical spinning particles in Cracow after 1947

$$i\partial_s\psi = \hat{H}(s)\psi, \quad \frac{d}{ds}(\psi|\chi) = 0$$

(but $\hat{H}(s)$ is not Hermitian with respect to the scalar product)

2. Transformation to the instant rest frame using Lorentz boost

$$s = s_R, \;\; z^i = h^i_k(s) z^k_R, \;\; (\psi_R|\chi_R) = \int d^3 z_R \, \psi^\dagger_R(s, \vec{z}_R)\chi_R(s, \vec{z}_R),$$

$$i\partial_s\psi_R = \hat{H}_R(s)\psi_R, \quad \hat{H}_R(s)^\dagger = \hat{H}_R(s)$$

3. The consistency conditions

$\psi(s, \vec{z})$: a wave packet (solution of the Dirac equation) centered on the line $X^\mu(s)$, i.e., such that for all s

$$(\psi|\hat{z}^i\psi) = 0.$$

Then

$$0 = -i\frac{d}{ds}(\psi_R|\hat{z}^i_R\psi_R) = (\psi_R|[\hat{H}_R, \hat{z}^i_R]\psi_R), \quad 0 = (\psi_R|[\hat{H}_R, [\hat{H}_R, \hat{z}^i_R]]\psi_R), \;\; \ldots$$

The commutators do not vanish. In consequence, we obtain nontrivial conditions on the expectation values of observables, and on $X^\mu(s)$.

Classical spinning particles in Cracow after 1947

Explicit form of $\psi(s,\vec{z})$ is not needed!

4. The classical equations of motion

The Foldy-Wouthuysen representation to the order m^{-2}, and all terms quadratic and higher in $F_{\mu\nu}$ neglected. In the case $g_0 = 2$:

$$m\ddot{X}_\mu = eF_{\mu\nu}\dot{X}^\nu + \frac{e}{2m}\epsilon_{\nu\lambda\sigma\alpha}\dot{X}^\lambda(\delta^\beta_\mu - \dot{X}^\beta\dot{X}_\mu)W^\sigma F^{\alpha\nu}_{,\beta} + \frac{e}{2m}(\delta^\sigma_\mu - \dot{X}^\sigma\dot{X}_\mu)F_{\nu\sigma,\rho}C^{\rho\nu} + \frac{e}{2m}\dot{X}^\rho\dot{X}_\nu F^{\nu\sigma}_{,\rho}C_{\mu\sigma}$$

$$\frac{dW^\lambda}{ds} = -\dot{X}^\lambda\ddot{X}_\mu W^\mu + \frac{e}{m}(\delta^\lambda_\mu - \dot{X}^\lambda\dot{X}_\mu)F^{\mu\nu}W_\nu + \frac{e}{m}(\delta^\lambda_\mu - \dot{X}^\lambda\dot{X}_\mu)F^{\mu\sigma,\rho}Z_{\sigma\rho} + \frac{1}{m}(\ddot{X}^\lambda P^\nu_\nu + \ddot{X}_\nu P^{\nu\lambda}) + \frac{e}{m^2}(\ldots)$$

W^σ – the spin 4-vector – is related to $(|\hat{\Sigma}^i_R)$, $\quad W^\mu\dot{X}_\mu \equiv 0$.
$C^{\rho\nu}(s) \sim (|\hat{\pi}^R_k\hat{z}^i_R + \hat{z}^i_R\hat{\pi}^R_k)$, $Z_{\rho\sigma}(s) \sim (|\hat{\Sigma}^k\hat{z}^i_R)$, $P^{\nu\lambda}(s) \sim (|\hat{\Sigma}^k\hat{\pi}^R_i)$,
where $\hat{\pi}^R_k = i\frac{\partial}{\partial z^k_R} - eB^R_k$, are classical dynamical variables too

Summary

1. Mathisson's paper *New Mechanics of Material Systems* offers the extremely interesting possibility of dimensional transfiguration of dynamical systems. I think it has not been explored to a satisfactory degree yet, especially beyond the theory of spinning matter. Also this aspect of that renowned paper should be emphasized, not only the derivation of equations of motion for the spinning particle.

2. The quantum mechanical expectation values can be regarded as a kind of moments. There is infinite number of them, because, in general, various degrees of freedom of the quantum particle are entangled: the expectation values of products of operators do not factorize. Are these moments related with the moments introduced in the way Mathisson did?

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THANK YOU!