Dark Energy & Modified Gravity with Higher-Order Scalar-Tensor theories

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Dark energy: introduction

- So far, **General Relativity** seems compatible with all observations.

- The cosmological acceleration can be accounted for by a (tiny) cosmological constant.

- Models of **dark energy** & **modified gravity:**
  quintessence, K-essence, f(R) gravity, massive gravity…

- Scalar-tensor theories with **higher order derivatives**

Scalar-tensor theories

- Simplest extension of GR: add a **scalar field**

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} \, {}^{(4)}R - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \right]$$

- K-essence:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} \, {}^{(4)}R + P(X, \phi) \right]$$

$$X \equiv \nabla_\mu \phi \, \nabla^\mu \phi$$

Higher order scalar-tensor theories

- Usual scalar-tensor theories : $\mathcal{L}(\nabla_\lambda \phi, \phi)$

- Generalized theories with second order derivatives

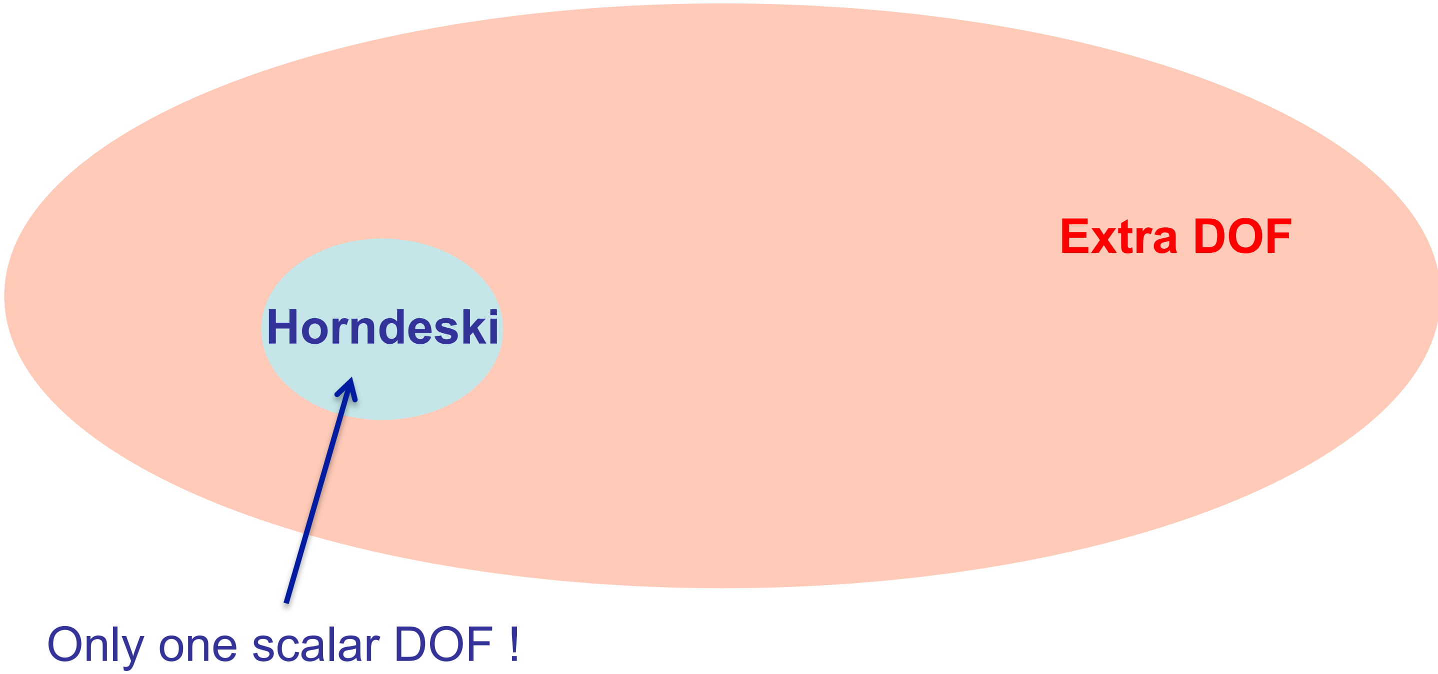
$$\mathcal{L}(\nabla_\mu \nabla_\nu \phi,\ \nabla_\lambda \phi,\ \phi)$$

- In general, they contain an **extra degree of freedom**, expected to lead to **Ostrogradsky instabilities**

$$L(\ddot{q}, \dot{q}, q)$$

Higher order scalar-tensor theories

- Usual theories (Brans-Dicke theories) $\mathcal{L}(\nabla_\lambda \phi, \phi)$
- Generalized theories: $\mathcal{L}(\nabla_\mu \nabla_\nu \phi,\ \nabla_\lambda \phi,\ \phi)$



Horndeski theories

Horndeski 74

- Combination of the following four Lagrangians

$$L_2^H = G_2(\phi, X)$$
$$L_3^H = G_3(\phi, X)\,\Box\phi$$
$$L_4^H = G_4(\phi, X)\,{}^{(4)}R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})$$
$$L_5^H = G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})$$

with
$$X \equiv \nabla_\mu\phi\nabla^\mu\phi$$
$$\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$$

- **Second order equations of motion** for the scalar field and the metric

- They contain 1 scalar DOF and 2 tensor DOF.

  No dangerous extra DOF !

Higher order scalar-tensor theories

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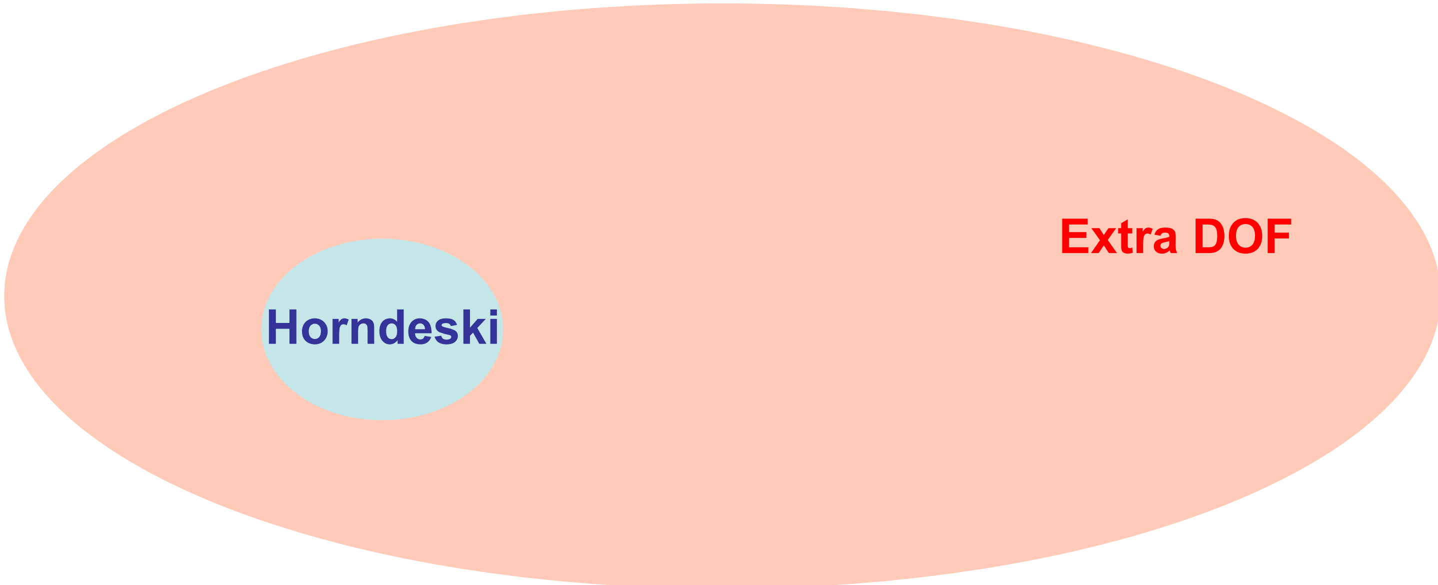

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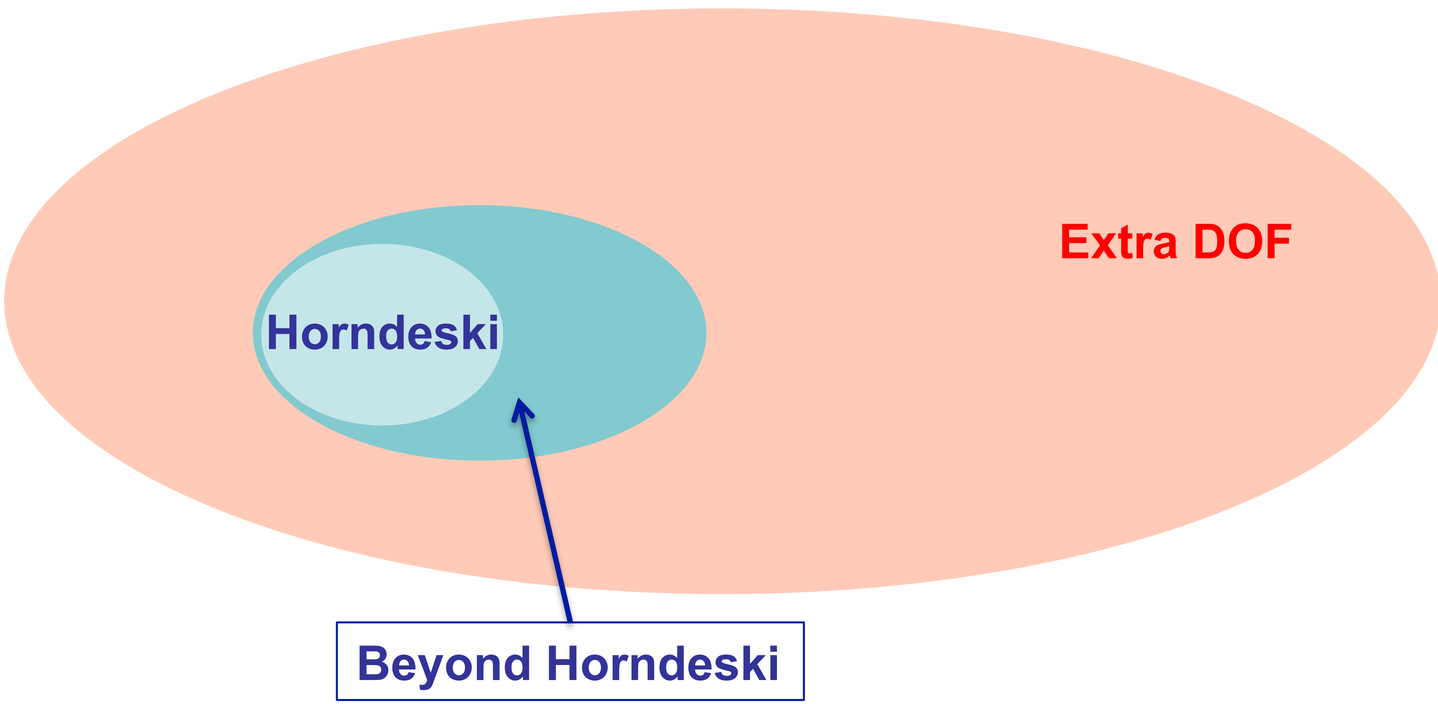

Higher order scalar-tensor theories

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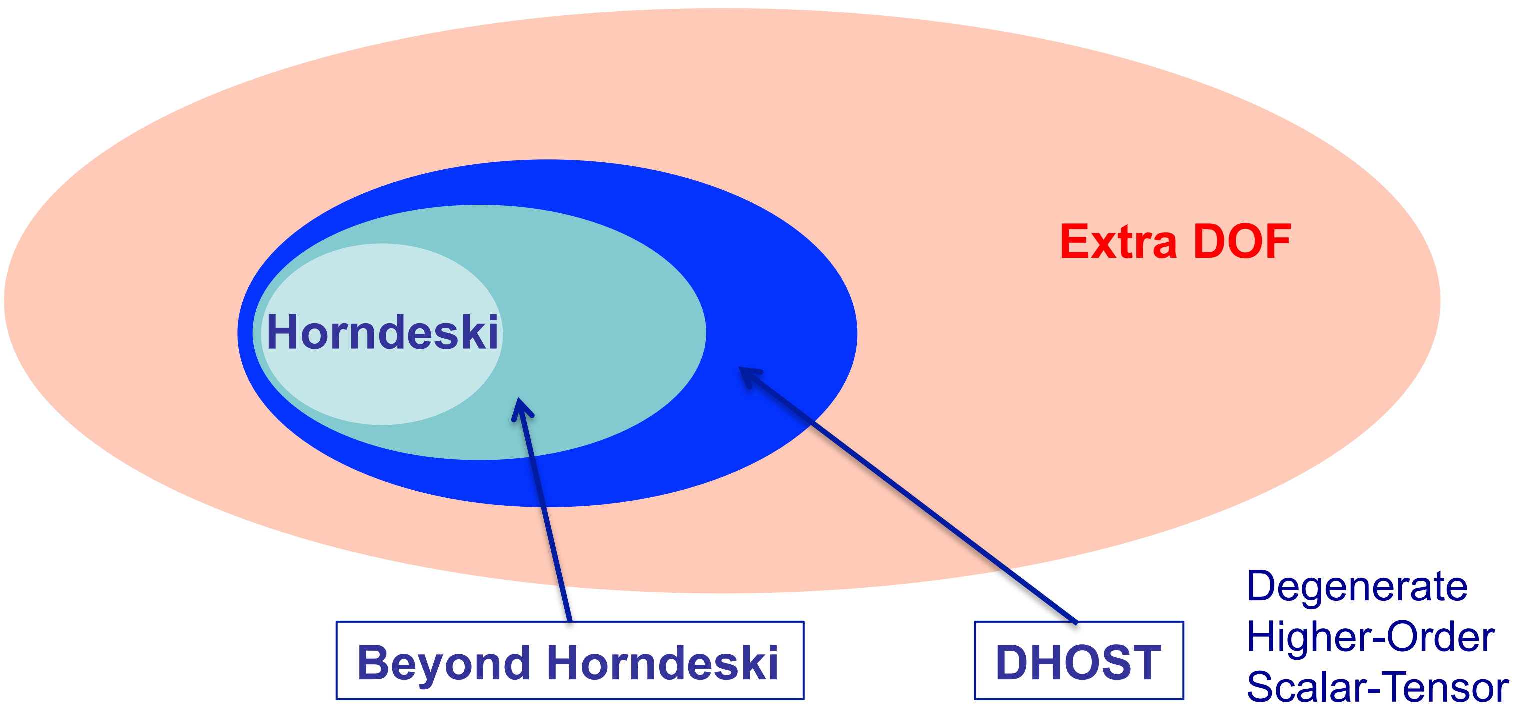


Beyond Horndeski

- First hint: disformal transformation of Einstein-Hilbert

Zumalacarregui & Garcia-Bellido '13

- Two extensions "beyond Horndeski"

[Gleyzes, DL, Piazza & Vernizzi '14]

$$L_4^{\rm bH} \equiv F_4(\phi, X)\, \epsilon^{\mu\nu\rho}{}_{\sigma}\, \epsilon^{\mu'\nu'\rho'\sigma} \phi_\mu \phi_{\mu'} \phi_{\nu\nu'} \phi_{\rho\rho'}$$

$$L_5^{\rm bH} \equiv F_5(\phi, X) \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma'} \phi_\mu \phi_{\mu'} \phi_{\nu\nu'} \phi_{\rho\rho'} \phi_{\sigma\sigma'}$$

leading to **third order** equations of motion.

- In contrast with earlier belief, no extra DOF if the total Lagrangian is "**degenerate**".

Presence of constraints in the system

Degenerate Lagrangians

DL & K. Noui ‘1510

- **Scalar-tensor theories**: scalar field + metric
- Simple toy model: $\phi(x^\lambda) \to \phi(t)\,, \quad g_{\mu\nu}(x^\lambda) \to q(t)$
- Lagrangian

$$L = \frac{1}{2} a\, \ddot{\phi}^2 + b\, \ddot{\phi}\, \dot{q} + \frac{1}{2} c\, \dot{q}^2 + \frac{1}{2} \dot{\phi}^2 - V(\phi, q)$$

- Equations of motion are higher order
  (4th order if a nonzero, 3rd order if a=0)

Degrees of freedom

- Introduce the auxiliary variable $Q \equiv \dot{\phi}$

$$L = \frac{1}{2} a \dot{Q}^2 + b \dot{Q} \dot{q} + \frac{1}{2} c \dot{q}^2 + \frac{1}{2} Q^2 - V(\phi, q) - \lambda (Q - \dot{\phi})$$

- Equations of motion

$$a \ddot{Q} + b \ddot{q} = Q - \lambda \qquad \dot{\phi} = Q \,, \quad \dot{\lambda} = -V_\phi$$

$$b \ddot{Q} + c \ddot{q} = -V_q$$

- If the Hessian matrix is **invertible**, one finds **3 DOF.**

$$M \equiv \left(\frac{\partial^2 L}{\partial v^a \partial v^b} \right) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

[6 initial conditions]

Degrees of freedom

- If the Hessian matrix is degenerate, i.e. $ac - b^2 = 0$ then only **2 DOF** (at most).

$$M \equiv \left(\frac{\partial^2 L}{\partial v^a \partial v^b} \right) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

[$\ddot{\phi}$ can be absorbed in $\dot{x} \equiv \dot{q} + \frac{b}{c}\ddot{\phi}$]

- Hamiltonian analysis: primary constraint and secondary constraint

[$p_a = \frac{\partial L}{\partial v^a}(v)$ cannot be inverted]

Generalization (classical mechanics)

[Motohashi, Noui, Suyama, Yamaguchi & DL 1603]

- Consider a general Lagrangian of the form

$$L(\ddot{\phi}^\alpha, \dot{\phi}^\alpha, \phi^\alpha; \dot{q}^i, q^i) \qquad \alpha = 1, \cdots, n; \; i = 1, \cdots, m$$

In general, one finds **2n+m dof**. But the n extra DOF can be eliminated by requiring:

1. **Primary conditions** (n **primary** constraints)

$$L_{\dot{Q}^\alpha \dot{Q}^\alpha} - L_{\dot{Q}^\alpha \dot{q}^i} (L^{-1})^{\dot{q}^i \dot{q}^j} L_{\dot{q}^j \dot{Q}^\beta} = 0$$

2. **Secondary conditions** (n **secondary** constraints)

$$L_{\dot{Q}^\alpha \dot{\phi}^\beta} - L_{\dot{Q}^\beta \dot{\phi}^\alpha} = 0 \qquad \text{if} \quad m = 0$$

[See also Klein & Roest 1604]

Higher Order Scalar-Tensor theories

- Consider all Lagrangians of the form

[DL & Noui '1510]

$$S[g,\phi] = \int d^4x\,\sqrt{-g}\,\left[f_2{}^{(4)}R + C_{(2)}^{\mu\nu\rho\sigma}\,\nabla_\mu\nabla_\nu\phi\,\nabla_\rho\nabla_\sigma\phi\right]$$

where $f_2 = f_2(X,\phi)$ and $C_{(2)}^{\mu\nu\rho\sigma}$ depends only on ϕ and $\nabla_\mu\phi$.

- Equivalently: $C_{(2)}^{\mu\nu\rho\sigma}\,\phi_{\mu\nu}\,\phi_{\rho\sigma} = \sum a_A(X,\phi)\,L_A^{(2)}$

$$L_1^{(2)} = \phi_{\mu\nu}\,\phi^{\mu\nu}\,, \quad L_2^{(2)} = (\Box\phi)^2\,, \quad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu$$
$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \quad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2$$

- **7 degenerate subclasses**: 4 with $f_2 \neq 0$, 3 with $f_2 = 0$

- They include $L_4^{\rm H}$ and $L_4^{\rm bH}$

[See also Crisostomi et al '1602; Ben Achour, DL & Noui '1602; de Rham & Matas '1604]

Disformal transformations

- Transformations of the metric [Bekenstein '93]

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(X,\phi)\, g_{\mu\nu} + D(X,\phi)\, \partial_\mu\phi\, \partial_\nu\phi$$

- Starting from an action $\tilde{S}\,[\phi, \tilde{g}_{\mu\nu}]$, one can define the new action

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}\,[\phi, \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\, \phi_\mu \phi_\nu]$$

- Disformal transformation of quadratic DHOST theories ?

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}}\left[\tilde{f}\,\tilde{R} + \tilde{\alpha}_I \tilde{L}_I^\phi\right]$$

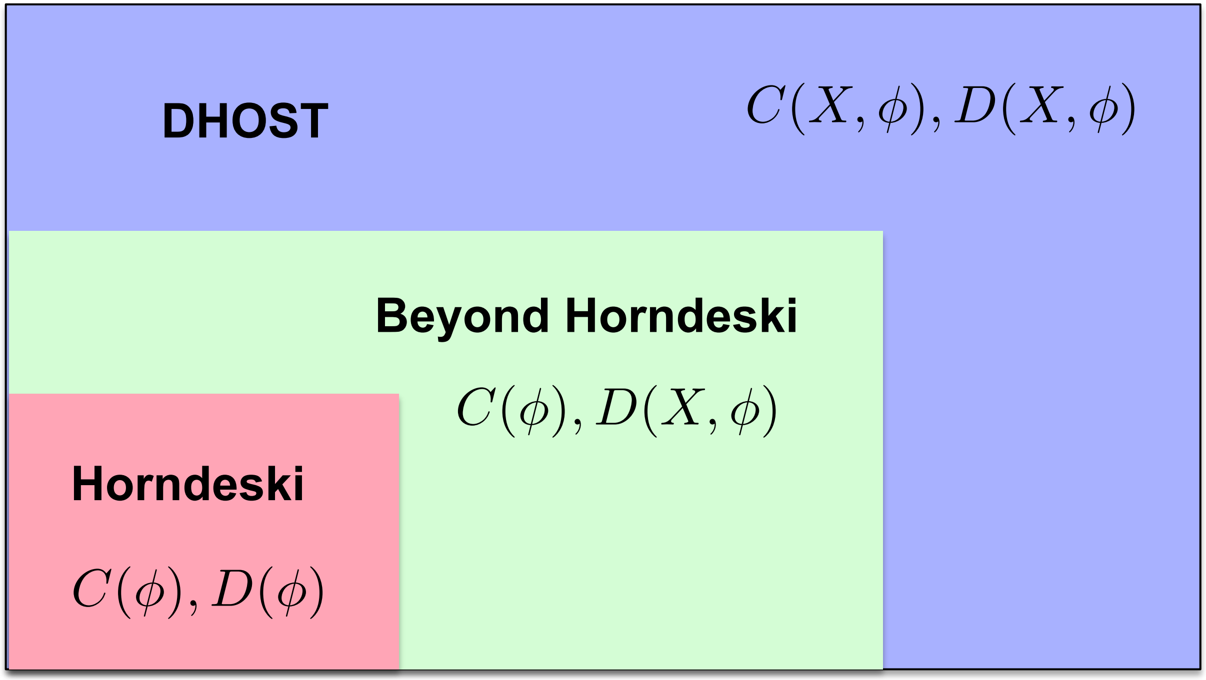
The structure of DHOST theories is preserved and all seven subclasses are stable.

[Ben Achour, DL & Noui '1602]

Disformal transformations

- Stability under the transformations:

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\, \partial_\mu\phi\, \partial_\nu\phi$$



Cubic DHOST theories

[Ben Achour, Crisostomi, Koyama, DL, Noui & Tasinato '1608]

- Action of the form

$$S[g,\phi] = \int d^4x \sqrt{-g} \left[f_3\, G^{\mu\nu} \phi_{\mu\nu} + C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\, \phi_{\mu\nu}\, \phi_{\rho\sigma}\, \phi_{\alpha\beta} \right]$$

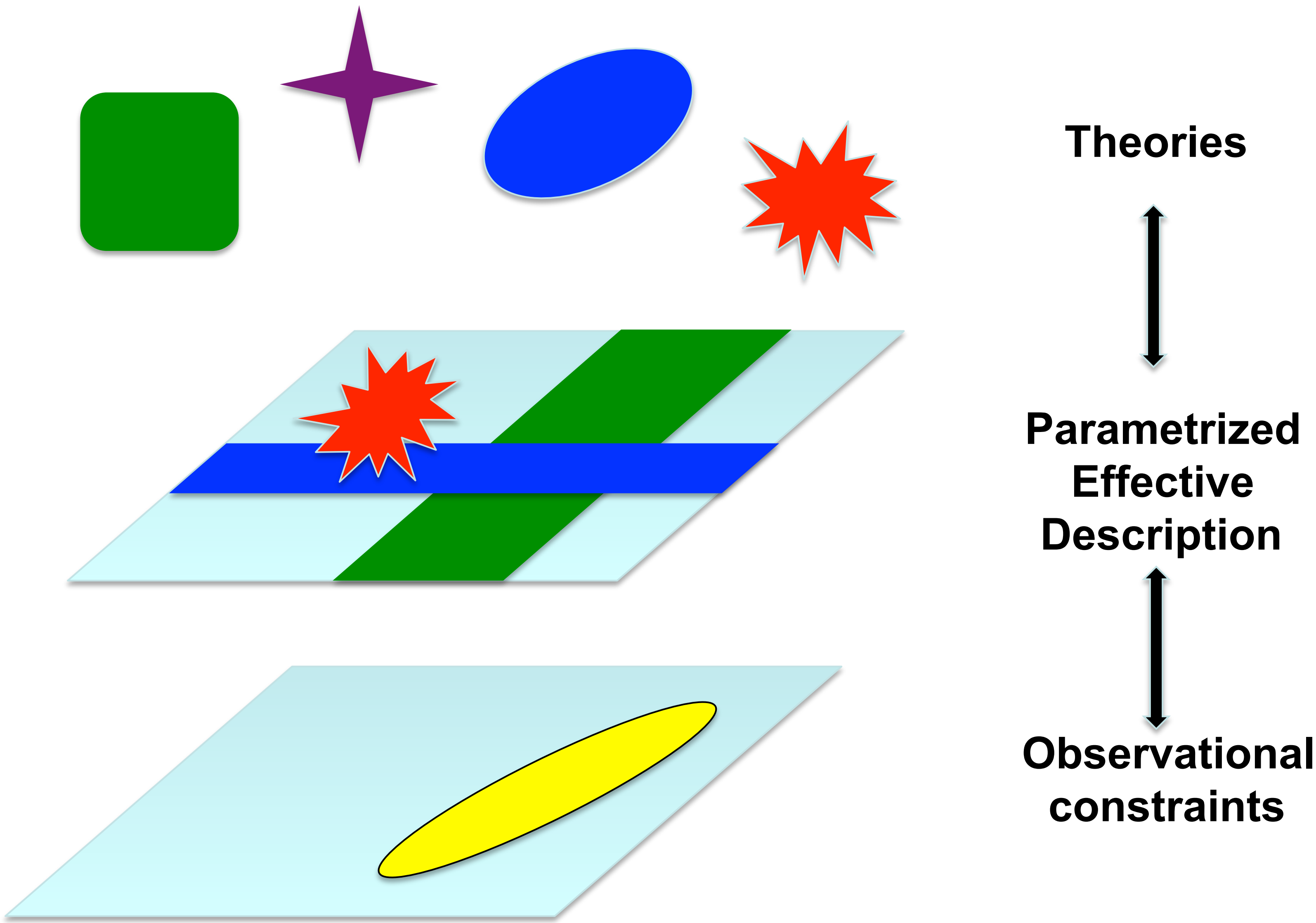
depends on eleven functions: $C_{(3)}^{\mu\nu\rho\sigma\alpha\beta}\, \phi_{\mu\nu}\, \phi_{\rho\sigma}\, \phi_{\alpha\beta} = \sum_{i=1}^{10} b_i(X,\phi)\, L_i^{(3)}$

- This includes the Lagrangians L_5^{H} and L_5^{bH} .

- 9 degenerate subclasses: 2 with $f_3 \neq 0$, 7 with $f_3 = 0$

- 25 **combinations of quadratic and cubic theories** (out of 7x9) are degenerate.

Cosmology:
Effective description of Dark Energy & Modified Gravity



Theories
Parametrized
Effective
Description
Observational
constraints

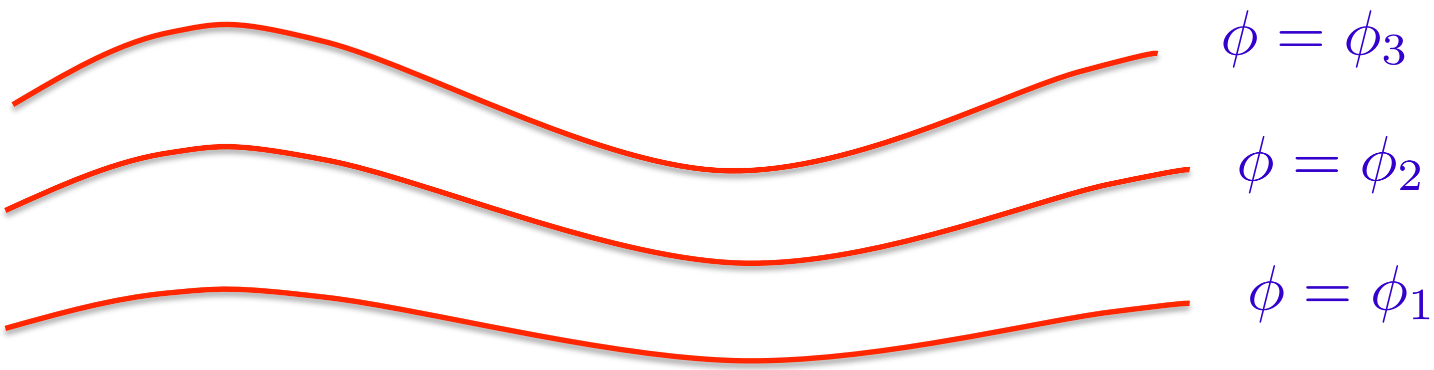
Effective description of Dark Energy

[See e.g review: Gleyzes, DL & Vernizzi 1411.3712]

- Restriction: **single scalar field** models

- The scalar field defines a **preferred slicing**

  Constant time hypersurfaces = uniform field hypersurfaces



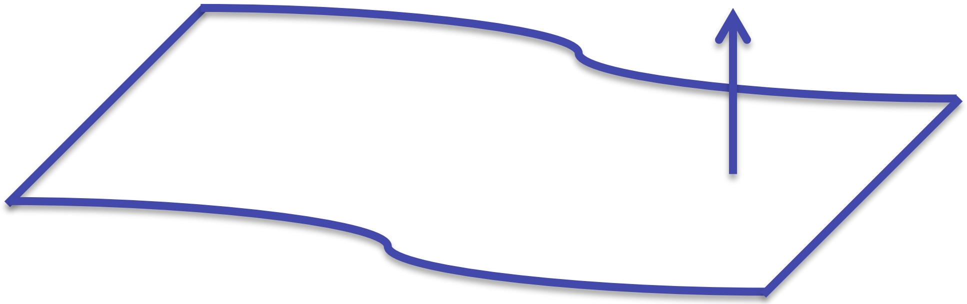
- All perturbations embodied by the metric only

Uniform scalar field slicing

- **3+1 decomposition** based on this preferred slicing

- Basic ingredients
  - **Unit vector normal** to the hypersurfaces

$$n^\mu = -\frac{\nabla^\mu \phi}{\sqrt{-(\nabla\phi)^2}}$$

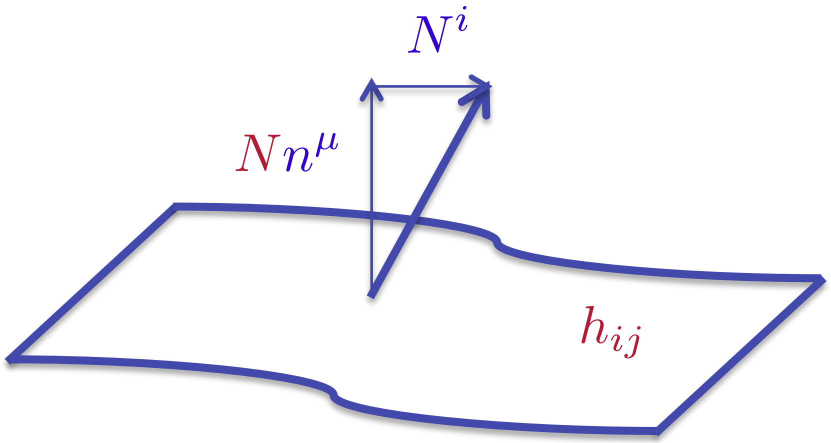


  - **Projection** on the hypersurfaces: $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

ADM formulation

- **ADM decomposition of spacetime**

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$



Extrinsic curvature:

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right)$$

Intrinsic curvature: R_{ij}

$$X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = -\frac{\dot{\phi}^2(t)}{N^2}$$

- **Generic Lagrangians of the form**

$$S_g = \int d^4x\, N\sqrt{h}\, L(N, K_{ij}, R_{ij}; t)$$

Homogeneous background & linear perturbations

- **Background** $ds^2 = -\bar{N}^2(t)\,dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j$

$$\bar{L}(a,\dot{a},\bar{N}) \equiv L\left[K^i_j = \frac{\dot{a}}{\bar{N}a}\,\delta^i_j, R^i_j = 0, N = \bar{N}(t)\right]$$

- **Perturbations:** $\delta N \equiv N - \bar{N}\,,\ \delta K^i_j \equiv K^i_j - H\delta^i_j\,,\ \delta R^i_j \equiv R^j_i$

- Expanding the Lagrangian $L(q_A)$ with $q_A \equiv \{N, K^i_j, R^i_j\}$

yields $$L(q_A) = \bar{L} + \frac{\partial L}{\partial q_A}\delta q^A + \frac{1}{2}\frac{\partial^2 L}{\partial q_A \partial q_B}\delta q_A \delta q_B + \ldots$$

- The **quadratic** action describes the **dynamics of linear perturbations**

Linear perturbations

- Quadratic action

Gleyzes, DL, Piazza & Vernizzi '13,
[notation: Bellini & Sawicki '14]

$$S^{(2)} = \int dx^3 dt\, a^3\, \frac{M^2}{2} \Big[\delta K^i_j \delta K^j_i - \delta K^2 + \alpha_K H^2 \delta N^2 + 4\, \alpha_B H\, \delta K\, \delta N + (1+\alpha_T) \delta_2 \left(\frac{\sqrt{h}}{a^3} R \right) + (1+\alpha_H) R\, \delta N \Big]$$

$$\alpha_M \equiv \frac{d \ln M^2}{H\, dt}$$

	α_K	α_B	α_M	α_T	α_H
Quintessence, K-essence	✓				
Kinetic braiding, DGP	✓	✓			
Brans-Dicke, f(R)	✓	✓	✓		
Horndeski	✓	✓	✓	✓	
Beyond Horndeski	✓	✓	✓	✓	✓

Scalar degree of freedom

- Scalar perturbations: $\delta N\,,\quad N_i \equiv \partial_i \psi\,,\quad h_{ij} = a^2(t) e^{2\zeta}\delta_{ij}$

- Quadratic action for the **physical degree of freedom**:

$$S^{(2)} = \frac{1}{2}\int dx^3 dt\, a^3 \left[\mathcal{K}_t\, \dot{\zeta}^2 + \mathcal{K}_s \frac{(\partial_i \zeta)^2}{a^2}\right]$$

$$\mathcal{K}_t \equiv \frac{\alpha_K + 6\alpha_B^2}{(1+\alpha_B)^2}\,, \quad \mathcal{K}_s \equiv 2M^2 \left\{1 + \alpha_T - \frac{1+\alpha_H}{1+\alpha_B}\left(1 + \alpha_M - \frac{\dot{H}}{H^2}\right) - \frac{1}{H}\frac{d}{dt}\left(\frac{1+\alpha_H}{1+\alpha_B}\right)\right\}$$

- Stability
  - No ghost: $\mathcal{K}_t > 0$
  - No gradient instability: $c_s^2 \equiv -\dfrac{\mathcal{K}_s}{\mathcal{K}_t} > 0$

Tensor degrees of freedom

- Quadratic action for the **tensor modes**:

$$S_\gamma^{(2)} = \frac{1}{2} \int dt\, d^3x\, a^3 \left[\frac{M^2}{4} \dot{\gamma}_{ij}^2 - \frac{M^2}{4} (1 + \alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right]$$

- Stability
  - No ghost: $M^2 > 0$
  - No gradient instability: $c_T^2 \equiv 1 + \alpha_T > 0$

Extension to DHOST theories

DL, Mancarella, Noui & Vernizzi '1703

- Quadratic action in terms of **9 functions of time**

$$S_{\rm quad} = \int d^3x\, dt\, a^3\, \frac{M^2}{2}\left\{\delta K_{ij}\delta K^{ij} - \left(1+\frac{2}{3}\alpha_{\rm L}\right)\delta K^2 + (1+\alpha_T)\left(R\frac{\delta\sqrt{h}}{a^3} + \delta_2 R\right)\right.$$
$$\left. + H^2\alpha_K \delta N^2 + 4H\alpha_B \delta K \delta N + (1+\alpha_H) R\delta N + 4\beta_1 \delta K \delta\dot{N} + \beta_2 \delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i \delta N)^2 \right\}$$

- **Degeneracy conditions:** 2 possible sets

$$\mathcal{C}_{\rm I}:\ \alpha_{\rm L} = 0\,,\ \beta_2 = -6\beta_1^2\,,\ \beta_3 = -2\beta_1\left[2(1+\alpha_{\rm H}) + \beta_1(1+\alpha_{\rm T})\right]$$

$$\mathcal{C}_{\rm II}:\ \beta_1 = -(1+\alpha_{\rm L})\frac{1+\alpha_{\rm H}}{1+\alpha_{\rm T}}\,,\ \beta_2 = -6(1+\alpha_{\rm L})\frac{(1+\alpha_{\rm H})^2}{(1+\alpha_{\rm T})^2}\,,\ \beta_3 = 2\frac{(1+\alpha_{\rm H})^2}{1+\alpha_{\rm T}}$$

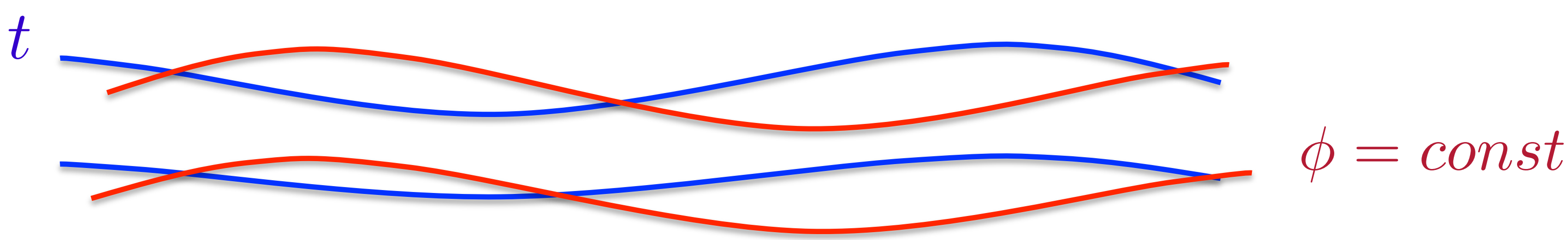
$\mathcal{C}_{\rm II}$: gradient instability either in the scalar or the tensor sector

Confrontation with observations

- Use a traditional gauge, e.g. Newtonian gauge

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)\,(1-2\Psi)\,\delta_{ij}\,dx^i dx^j$$

- Description in an arbitrary slicing ?



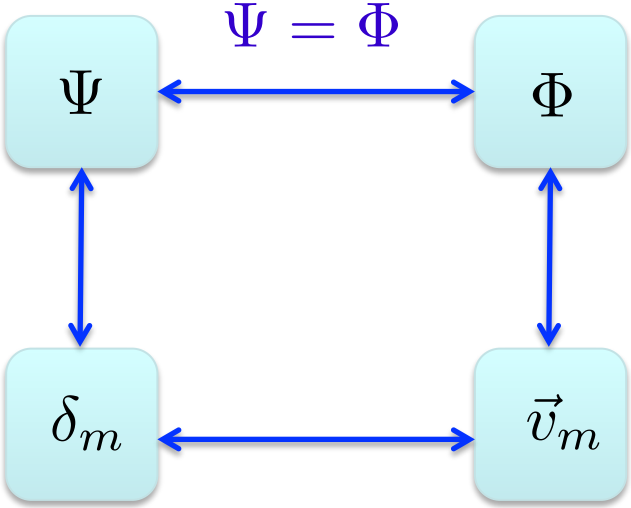
- Coordinate change $\quad t \to t + \pi(t, \vec{x})$

- Perturbations: $\quad \Phi\,,\ \ \Psi\,,\ \ \pi\,, \delta_m\,,\ \ \vec{v}_m$

Cosmological perturbations

- **Standard equations** (in GR)

$$\Psi = \Phi$$



$$\nabla^2\Psi = 4\pi G a^2 \rho_m \delta_m$$

(relativistic Poisson)

$$\dot{\vec{v}}_m + H\,\vec{v}_m = -\vec{\nabla}\Phi$$

(Euler)

$$\dot{\delta}_m + \vec{\nabla}\cdot\vec{v}_m = 0$$

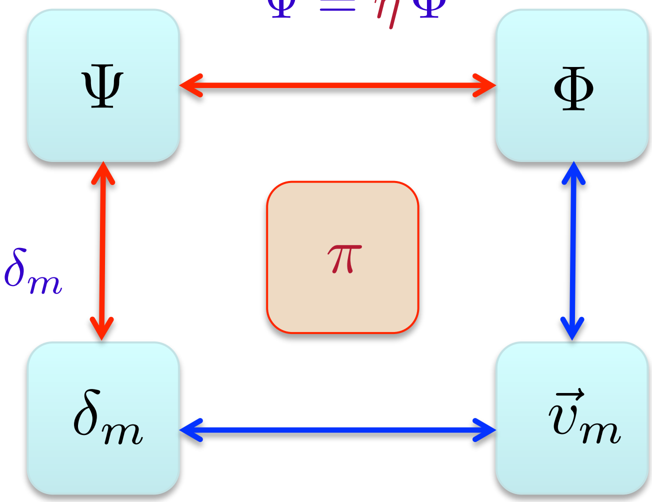
(continuity)

Cosmological perturbations

- **Modified equations**

$$\Psi = \eta\,\Phi$$

$$\nabla^2\Psi = 4\pi G(1+\Upsilon_G)a^2\rho_m\delta_m$$



Quasi-static approximation
(valid on scales
$kc_s \gg aH$
[Sawicki & Bellini '15])

$$G_{\rm eff} = G_{\rm eff}(\alpha_i)\,, \qquad \eta = \eta(\alpha_i)$$

which can be confronted to observations (galaxy clustering, weak lensing…)

[Bellini et al '15, Peronon et al. 15, Gleyzes et al 15, D'Amico et al 16, Alonso et al 16, …]

Neutron stars
in Modified Gravity

Stars in beyond Horndeski theories

Saito, Yamauchi, Mizuno, Gleyzes & DL ‘15
(see also Koyama & Sakstein ‘15)

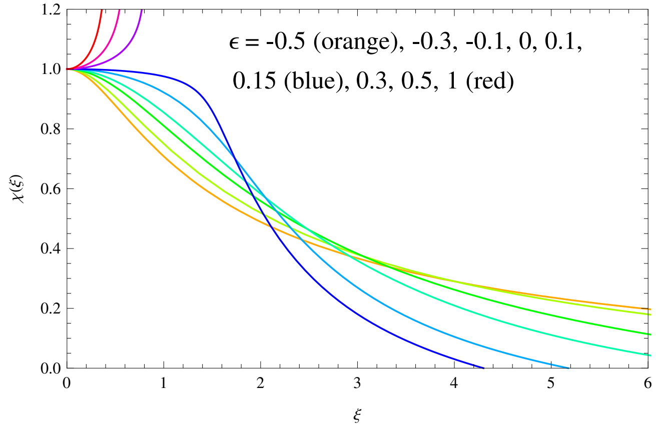
- Partial breaking of Vainshtein mechanism inside matter

Kobayashi, Watanabe & Yamauchi ‘14

- Spherical symmetry & nonrelativistic limit:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = G_{\mathrm{N}} \left(\frac{\mathcal{M}}{r^2} - \epsilon \frac{\mathrm{d}^2 \mathcal{M}}{\mathrm{d}r^2} \right) , \qquad \mathcal{M}(r) = 4\pi \int_0^r r'^2 \rho(r') \mathrm{d}r'$$

- Modified **Lane-Emden equation**
  (for $P = K\rho^{1+\frac{1}{n}}$)
  - Universal bound $\epsilon < 1/6$
  - Astrophysical constraints on $\Upsilon \equiv -4\epsilon$



$-0.18 < \Upsilon < 0.027$ [Sakstein 15, Jain et al 15]

Neutron stars in beyond Horndeski

Babichev, Koyama, DL, Saito & Sakstein '16

- Model

$$S = \int d^4x \sqrt{-g} \left[M_P^2 \left(\frac{R}{2} - \Lambda \right) - k_2 X + f_4 L_4^{\rm bH} \right]$$

with

$$L_4^{\rm bH} = -X \left[(\Box\phi)^2 - (\phi_{\mu\nu})^2 \right] + 2\phi^\mu \phi^\nu \left[\phi_{\mu\nu} \Box\phi - \phi_{\mu\sigma} \phi^\sigma{}_\nu \right]$$

- Cosmological solution: **de Sitter** with $\dot\phi = v_0 \neq 0 \,, \quad H \neq 0$

$$ds^2 = -(1 - H^2 r^2)\, dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega_2^2$$

$$\phi(r, t) = v_0 t + \frac{v_0}{2H} \ln\left(1 - H^2 r^2\right)$$

Neutron stars in beyond Horndeski

Babichev, Koyama, DL, Saito & Sakstein '16

- Spherical symmetric solutions

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega_2^2$$

with $\nu(r) = \nu_{\rm cosmo} + \delta\nu(r)\,, \quad \lambda(r) = \lambda_{\rm cosmo} + \delta\lambda(r)$

$$\phi(t,r) = \phi_{\rm cosmo}(t,r) + \delta\phi(r)$$

- **External solution**: Schwarzschild-de Sitter

$$ds^2 = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega_2^2\,, \qquad f \equiv 1 - \frac{2G_N M}{r} - H^2 r^2$$

$$\phi(t,r) = v_0\left[t - \int dr \frac{\sqrt{1-f}}{f}\right] \qquad G_N \equiv \frac{3G}{5\sigma^2 - 2}$$

Neutron stars in beyond Horndeski

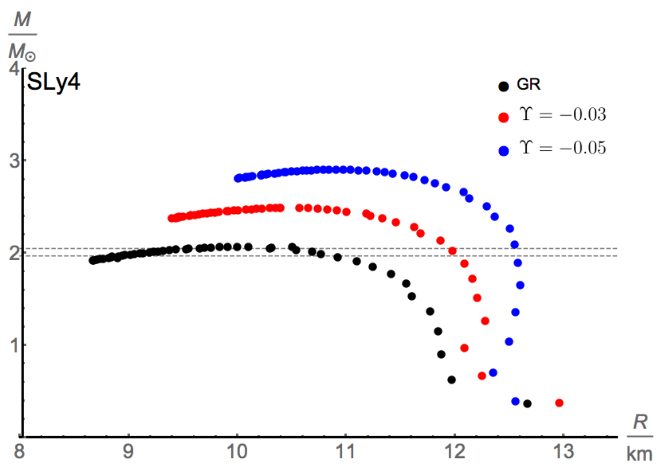
Babichev, Koyama, DL, Saito & Sakstein '16

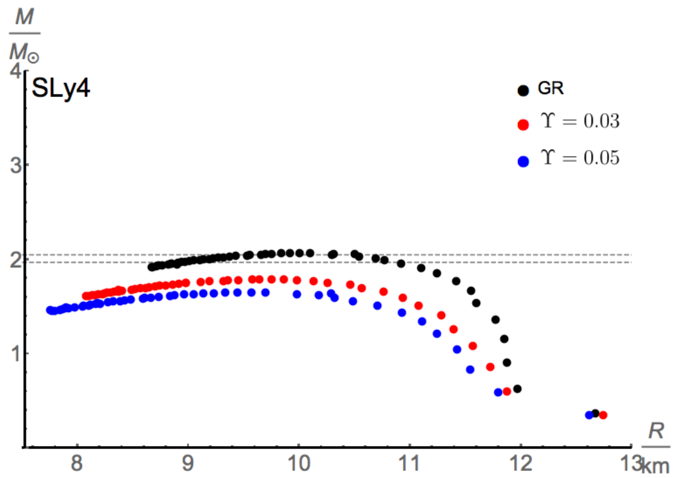
- Internal solution

  System analog to TOV equations

- Mass-radius relations

  For $\Upsilon < 0$ the maximum mass is larger than in GR.





- See also Sakstein, Babichev, Koyama, DL & Saito '16

Conclusions

- **DHOST theories:** systematic classification of "**degenerate**" theories that contain a single scalar DOF. They include and extend Horndeski and "beyond Horndeski" theories.

- **General disformal transformations** preserve all subclasses of (quadratic) DHOST theories.

- These theories of modified gravity can be tested in cosmology, by using the effective description of dark energy and modified gravity.

- Construction of Newtonian stars and neutron stars in simple beyond Horndeski models.