MATHISSON'S EQUATIONS WITHOUT HIM

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80 years ago Myron Mathisson published his known paper Neue mechanik materieller systeme. Acta Phys. Pol. Vol. 6 , 163 (1937) (received on September, 8, 1937); Translation in English: GRG, Vol. 42, 1011 (2010).

M. Mathisson (14.12 1897, Warsaw – 13.09 1940, Cambridge)

INTERNATIONAL CONFERENCE DEVOTED TO MYRON MATHISSON: HIS LIFE, WORK, AND INFLUENCE ON CURRENT RESEARCH (Wsrsaw, October 18–20, 2007)

Acta Phys. Pol. B. Proc. Suppl. vol. 1, (2008).

Neue Mechanik materieller Systeme

Nowa mechanika systemów materialnych

Von MYRON MATHISSON, Warschau

(Eingegangen am 8. September 1937)

Ausserhalb der Materie gelten die Feldgleichungen (= elektromagnetische und Gravitationsgleichungen) *der leeren Welt.* Diese Forderung fassen wir in die Form einer *Variationsgleichung* (§§ 1,2). Dabei lösen wir das materielle System in eine Summe von *Multipolen* auf. Im Falle der Gravitationsgleichungen wird man auf diese Weise auf den Begriff des Gravitationsskeletts geführt (§ 2). Im Gravitationsskelett ist der Pol für die Masse verantwortlich, der Dipol und der Quadrupol für den Drehimpuls (§§ 2, 3, 4, 5). Diese Zuordnung ergibt sich auf verschiedene Weisen, indem wir dem materiellen System bald eine aktive Rolle zuweisen (feld*erzeugende* Massen), bald eine passive (durch äusseres Feld angegriffene Massen). Nach Entwicklung einer Behandlungsmethode für unsere Variationsgleichung, gewinnen wir aus ihr die dynamischen Gesetze, welchen die Bestimmungsstücke unseres Gravitationsskeletts gehorchen müssen. So gelangen wir zwangsläufig zu mechanischen Gleichungen, die, im Vergleich mit den klassischen, neue Glieder enthalten. Neben Gliedern, die nur in äusseren Gravitationsfeldern zur Geltung kommen, erhalten wir ein neues Glied von grösster physikalischer Bedeutung, das an äussere Gravitationsfelder nicht gebunden ist.

Bei der Behandlung des Bewegungsproblems zeigt sich, dass der Drehimpuls als antisymmetrischer Tensor eingeführt werden muss. Der Begriff „Rotationsachse eines starren Körpers" wird nachträglich aus dem Drehimpuls konstruiert (§ 4). Bewegung des Schwerpunkts und Rotation sind miteinander gekoppelt. Der FOKKERsche Ansatz für die Bewegung der Achse eines symmetrischen Kreisels ist unhaltbar (§ 4). Will man das Gravitationsskelett zum vollständigen Gegenstück des klassischen dynamischen Modells eines Körpers ausbilden, so muss man neben dem Dipol einen Quadrupol einführen (§ 5). Die daraus entspringenden Gleichungen enthalten, als Sonderfall, die Präzessionsgleichungen, ein Umstand, der eine Verifikation unserer Ansätze gestattet. (§ 5, Ende). Berücksichtigung des Dipolglieds allein würde in einer Beschränkung auf den Fall der Trägheits*kugel* ihr klassisches Analogon finden.

Die neuen mechanischen Gleichungen lassen einen *Energiesatz* zu. Doch kommt eine neue Art Energie hinzu, *die Beschleunigungsenergie* (§ 6, Ende).

Dr. M. Mathisson

The death of Dr. Myron Mathisson on September 13 at the early age of forty-three has cut short an interesting line of research. Mathisson had been engaged for many years in studying the general dynamical laws governing the motion of a particle, with possibly a spin or a moment, in a gravitational or electromagnetic field, and had developed a powerful method of his own for passing from field equations to particle equations. The subject is of particular interest at the present time, as it has now become clear that quantum mechanics cannot solve the difficulties that arise in connexion with the interaction of point particles with fields, and a deeper classical analysis of the problem is needed. It is much to be regretted that Mathisson's death has occurred before the relations between his method and those of other workers on the subject have been completely elucidated.

Mathisson carried out his work at the Universities of Warsaw and Kazan and at an institute which he started in Cracow, and, since the spring of 1939, at Cambridge.

P. A. M. Dirac.

The Mathisson eqs. can be written as

$$\frac{D}{ds}\left(mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma} R^\lambda_{\pi\rho\sigma}, \tag{1}$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \tag{2}$$

where $u^\lambda \equiv dx^\lambda/ds$ is the particle's 4-velocity, $S^{\lambda\mu}$ is the tensor of spin, D/ds is the covariant derivative with respect to the particle's proper time s; $R^\lambda_{\pi\rho\sigma}$ is the Riemann curvature tensor (units $c = G = 1$ are used). The natural relationship for the tensor of spin used by Mathisson:

$$S^{\lambda\nu} u_\nu = 0 \tag{3}$$

Some stages of investigations after Mathisson

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$$S^{\lambda\nu} P_{\nu} = 0. \tag{4}$$

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About the Mathisson supplementary condition: "... arises in a natural fashion in the course of the derivation".

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Mathisson's eqs. in the comoving tetrad representation

It follows from the propagation set of Mathisson's equations:

$$m\gamma_{(1)(4)(4)} = S_{(1)} R_{(1)(4)(2)(3)}, \tag{5}$$

$$m\gamma_{(2)(4)(4)} = S_{(1)} \big(R_{(2)(4)(2)(3)} - \dot{\gamma}_{(3)(4)(4)} - \gamma_{(2)(4)(4)} \gamma_{(2)(3)(4)} \big), \tag{6}$$

$$m\gamma_{(3)(4)(4)} = S_{(1)} \big(R_{(3)(4)(2)(3)} + \dot{\gamma}_{(2)(4)(4)} - \gamma_{(3)(4)(4)} \gamma_{(2)(3)(4)} \big). \tag{7}$$

The consequence of the second set of Mathisson's equations:

$$\gamma_{(i)(k)(4)} = 0. \tag{8}$$

$$\gamma_{(i)(4)(4)} = a_{(i)}. \tag{9}$$

There is the relationship following from this representation in the linear spin approximation for any metric [R. Plyatsko, Phys. Rev. D, vol. 58, 084031 (1998)]:

$$a_{(i)} = \frac{S_{(1)}}{m} R_{(i)(4)(2)(3)} = \frac{S_{(1)}}{m} B^{(1)}_{(i)}, \tag{10}$$

where $a_{(i)}$ are the local components of the particle 3-acceleration relative to geodesic free fall as measured by the comoving observer; $S_{(1)}$ is the single nonzero component of the particle spin. Gravitomagnetic components by [K. Thorne, J. Hartle, Phys. Rev. D, vol. 31, 1815 (1985)]:

$$B^{(i)}_{(k)} = -\frac{1}{2} R^{(i)(4)}{}_{(m)(n)} \varepsilon^{(m)(n)}{}_{(k)}. \tag{11}$$

Condition for a test particle:

$$\frac{|S_0|}{mr} \equiv \varepsilon \ll 1, \tag{12}$$

where $|S_0|$ is the absolute value of the particle's spin.

The first case:

$$|\vec{a}| = \frac{3M}{r^2}\varepsilon\delta, \tag{13}$$

where $\delta \equiv |u_\perp| \ll 1$. Note that M/r^2 numerically is equal to the Newtonian acceleration of free fall which is caused by a body with the mass M. The acceleration of a spinning particle is much less than the M/r^2.

Equatorial motions in Schwarzschild's background. Accrleration at high velocity

The second case ($u_\perp \gg 1$):

$$|\vec{a}| = \frac{3M}{r^2}\varepsilon\gamma^2, \tag{14}$$

where γ is the Lorentz factor calculated by the tangential velocity $u_\perp$. Expression (14) shows that for any small value ε one can choose such the high values γ which would lead to $|\vec{a}| \gg M/r^2$.

So, according to the Mathisson eqs., from the point of view of the observer comoving with a spinning particle in Schwarzschild's background, the spin-gravity interaction is much greater at the highly relativistic particle's velocity than at the low velocity.

Strong repulsive action of spin-gravity coupling: highly relativistic circular orbits

It is known that according to geodesic equations a spinless test particle with nonzero mass can move on circular orbits in Schwarzschild's background only for $r > 1.5r_g$.

The Mathisson equations admit highly relativistic circular orbits of a spinning particle in Schwarzschild's background for $r = 1.5r_g$ and in the space region $r_g < r < 1.5r_g$ because of the significant repulsive action of the spin-gravity coupling. The necessary values of the particle orbital velocity for the motions on these orbits correspond to the relativistic γ-factor of the order $1/\sqrt{\varepsilon}$.

Strong spin-gravity attractive action on the highly relativistic circular orbits in the space regin $r > 1.5r_g$.

Numerical estimates

By the numerical estimates for an electron in the gravitational field of a black hole with three of the Sun's mass the necessary value of the γ-factor for the realization of some highly relativistic orbits by the electron near this black hole is of order 10^8. This γ-factor corresponds to the energy of the electron free motion of order 10^{14} eV. Analogously, for a proton in the field of such a black hole the corresponding energy is of order 10^{18} eV. For the massive black hole those values are greater: for example, if M is equal to 10^6 of the Sun's mass the corresponding value of the energy for an electron is of order 10^{17} eV and for a proton it is 10^{21} eV.

For a neutrino near the black hole with three of the Sun's mass the necessary values of its γ-factor for motions on the highly relativistic circular orbits correspond to the neutrino's energy of the free motion of order 10^5 eV. If the black hole's mass is of order 10^6 of the Sun's mass, the corresponding value is of order 10^8 eV.

Some publications

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Conclusions

The Mathisson equations are an important source of knowledge about basic properties of gravitational interactions in the highly relativistic region.

Highly relativistic motions of a spinning particle in the Schwarzschild, Schwarzschild–de Sitter, and Kerr backgrounds give the new theoretical data concerning physical effects following from general relativity. At the same time, it is useful to take into account the corresponding results in the practical high energy physics, astrophysics, and cosmology.

THANK YOU!