Properties of null one-way solution with separated variables of Maxwell equations in Kerr space-time

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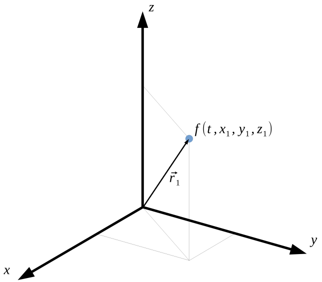
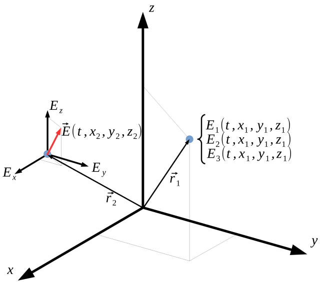
26.09.2017

Scalar field



Vector field



$$f_{tt} - \Delta f = 0. \qquad (1)$$

$$\begin{aligned} \frac{\partial^2 E_1}{\partial t^2} - \Delta E_1 &= 0, \\ \frac{\partial^2 E_2}{\partial t^2} - \Delta E_2 &= 0, \\ \frac{\partial^2 E_3}{\partial t^2} - \Delta E_3 &= 0. \end{aligned} \qquad (2)$$

PDE in spherical coordinate system

- Square of line element in spherical coordinate system is

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{3}$$

- Scalar wave equation — single equation

$$\frac{\partial^2 f}{\partial t^2} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right) = 0. \tag{4}$$

- Vector wave equation — coupled system of equations

$$\begin{cases} \frac{\partial^2 E_r}{\partial t^2} - c^2\left(\Delta E_r - 2\frac{E_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(E_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right) = 0, \\ \frac{\partial^2 E_\theta}{\partial t^2} - c^2\left(\Delta E_\theta - \frac{E_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial E_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\phi}{\partial\phi}\right) = 0, \\ \frac{\partial^2 E_\phi}{\partial t^2} - c^2\left(\Delta E_\phi - \frac{E_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial E_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\theta}{\partial\phi}\right) = 0. \end{cases} \tag{5}$$

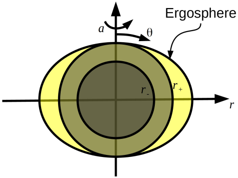
$\vec{E}(t,r,\theta,\phi) = E_r(t,r,\theta,\phi)\hat{r} + E_\theta(t,r,\theta,\phi)\hat{\theta} + E_\phi(t,r,\theta,\phi)\hat{\phi}$ — electric field vector, c— speed of light in vacuum.

The Kerr space-time

Rotating black hole space-time is described by Kerr metric, the square of line element in Boyer-Lindquist coordinates is (geometrized units are used $c = G = 1$, G — gravitational constant)

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mra\sin^2\theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \\ - \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \quad (6)$$

M — black hole mass parameter; a — angular momentum per unit mass; $\Sigma = r^2 + a^2\cos^2\theta$; $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, where $r_+ = M + \sqrt{M^2 - a^2}$, $r_- = M - \sqrt{M^2 - a^2}$. r_+ та r_- are event horizon and Cauchy horizon respectively.



- In 1972 Teukolsky (Phys. Rev. Lett. **29** 1972) by using NP approach has obtained decoupled equations for φ_2 and φ_0 in Petrov type D space-time

$$D\Delta\varphi_2 - \delta\bar{\delta}\varphi_2 + (2\gamma - \mu)D\varphi_2 + (2\alpha - \pi)\bar{\delta}\varphi_2 - (2\rho + \bar{\rho} + \epsilon - \bar{\epsilon})\,\Delta\varphi_2 +$$
$$+ (2\tau - \bar{\pi} + \bar{\alpha} + \beta)\,\delta\varphi_2 + (2\,(2\rho + \bar{\rho} + \epsilon - \bar{\epsilon})\,\gamma - 2\,(2\tau - \bar{\pi} + \bar{\alpha} + \beta) +$$
$$+2\,(\beta + \tau)\,\pi - 2\,(\epsilon + \rho)\,\mu + 2\delta\alpha - 2D\gamma + \Psi_2)\,\varphi_2 = 0; \quad (7)$$

$\varphi_{AB} = \varphi_2 o_A o_B - \varphi_1 o_A \iota_B - \varphi_1 \iota_A o_B + \varphi_0 \iota_A \iota_B$ — Maxwell spinor, $\varphi_2\colon \varphi_2 \mapsto \mathbb{C}$, $\varphi_1\colon \varphi_1 \mapsto \mathbb{C}$, $\varphi_0\colon \varphi_0 \mapsto \mathbb{C}$ — components of φ_{AB} in sninor dyad, o_A, ι_A basis spinors, $o_A\iota^A = 1$, $\iota_A o^A = -1$, $\iota_A\iota^A = o_A o^A = 0$;

$$\begin{aligned}
D\psi &= \frac{r^2+a^2}{r^2-2Mr+a^2}\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial r} + \frac{a}{r^2-2Mr+a^2}\frac{\partial\psi}{\partial\phi},\\
\Delta\psi &= \frac{1}{2(r^2+a^2\cos^2\theta)}\left((r^2+a^2)\frac{\partial\psi}{\partial t} - (r^2 - 2Mr + a^2)\frac{\partial\psi}{\partial r} + a\frac{\partial\psi}{\partial\phi}\right),\\
\delta\psi &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right),\\
\bar{\delta}\psi &= \frac{1}{\sqrt{2}(r-ia\cos\theta)}\left(-ia\sin\theta\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right).
\end{aligned} \quad (8)$$

Decoupling of equations in Newman-Penrose approach

- $\rho=-\dfrac{1}{r-ia\cos\theta},\ \mu=-\dfrac{r^2-2Mr+a^2}{2(r+ia\cos\theta)(r-ia\cos\theta)^2},$
  $\gamma=-\dfrac{(r^2-2Mr+a^2)(r+ia\cos\theta)-(r-M)(r^2+a^2\cos^2\theta)}{2(r+ia\cos\theta)^2(r-ia\cos\theta)^2},$
  $\alpha=-\dfrac{(r+ia\cos\theta)\cos\theta-2ia}{2\sqrt{2}\sin\theta(r-ia\cos\theta)^2},\beta=\dfrac{\cos\theta}{2\sqrt{2}\sin\theta(r+ia\cos\theta)},\tau=$
  $-\dfrac{ia\sin\theta}{\sqrt{2}(r+ia\cos\theta)(r-ia\cos\theta)},\ \pi=\dfrac{ia\sin\theta}{\sqrt{2}(r-ia\cos\theta)^2}$ — NP scalars,
  $\Psi_2=-\dfrac{M}{(r-ia\cos\theta)^3}$ — component of Weyl spinor.
- In the Kerr space-time in Boyer-Lindquist coordinates decoupled equation (7) is

$$\left(\frac{(r^2+a^2)^2}{r^2-2Mr+a^2}-a^2\sin^2(\theta)\right)\frac{\partial^2\psi}{\partial t^2}-(r^2-2Mr+a^2)\frac{\partial^2\psi}{\partial r^2}+\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right)-$$
$$-\left(\frac{1}{\sin^2(\theta)}-\frac{a^2}{r^2-2Mr+a^2}\right)\frac{\partial^2\psi}{\partial\phi^2}+\frac{4Mar}{r^2-2Mr+a^2}\frac{\partial^2\psi}{\partial t\partial\phi}+2\left(\frac{a(r-M)}{r^2-2Mr+a^2}-\frac{i\cos(\theta)}{\sin^2(\theta)}\right)\frac{\partial\psi}{\partial\phi}+$$
$$+2\left(\frac{M(r^2-a^2)}{r^2-2Mr+a^2}-r-ia\cos(\theta)\right)\frac{\partial\psi}{\partial t}+(\mathrm{ctg}^2(\theta)+1)\psi=0,\quad(9)$$

$$\psi=\rho^{-2}\varphi_2=(r-ia\cos\theta)^2\varphi_2.$$

Separation of variables

Teukolsky use the method of separation of variables in the form:

$$\psi = e^{-i\omega t} e^{im\phi} S(\theta) R(r), \tag{10}$$

which gives following ODE's (equation for φ_2)

$$(r^2 - 2Mr + a^2)\frac{d}{dr}\left(\frac{dR}{dr}\right) + \left(\frac{K^2+2i(r-M)K}{r^2-2Mr+a^2} - 4i\omega r - \lambda\right) R = 0, \tag{11}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + 2a\omega\cos\theta + \frac{2m\cos\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} + A\right) S = 0, \tag{12}$$

«where $K = (r^2 + a^2)\omega - am$ and $\lambda = A + a^2\omega^2 - 2am\omega$. Equation (12), together with boundary conditions of regularity at $\theta = 0$ and π, constitutes a Sturm-Liouville eigenvalue problem for the separation constant $A =_s A^m{}_l(a\omega)$. For fixed s, m, and $a\omega$, eigenvalues is labeled by l. The smallest eigenvalue is $l = max(|m|, |s|)$. From Sturm-Liouville theory, the eigenfunctions ${}_sS_l^m$ are complete and orthogonal on $0 \leq \theta \leq \pi$ for each m, s, and $a\omega$. When $s = 0$, the eigenfunctions are the spheroidal wave functions $S_l^m(-a^2\omega^2, \cos\theta)$ (cf. Flammer 1957). When $a\omega = 0$, the eigenfunctions are the spin-weighted spherical harmonics ${}_sY^m{}_l =_s S^m{}_l(\theta)e^{im\phi}$, and $A = (l - s)(l + s + 1)$ (cf. Goldberg et al. 1967). In general case, we shall refer to the eigenfuctions as **"spin-weighted spheroidal harmonics"**. Numerical calculation of these functions and the corresponding eigenvalues is described in Paper II».

One-way null Maxwell field

- For obtaining solution in analytical form we consider algebraically special Maxwell field (Wald 1973, Chandrasekhar 1984).
- Let us choose the spin basis so that principal spinors α_A та β_B of the Maxwell spinor $\varphi_{AB} = \alpha_{(A}\beta_{B)}$ are proportional to the principal spinor $\gamma_A = \gamma_1 o_A$ of the Weyl spinor $\psi_{ABCD} = \gamma_{(A}\gamma_B\delta_C\delta_{D)}$. Then spinor φ_{AB} takes the form

$$\varphi_{AB} = \varphi_2 o_A o_B. \tag{13}$$

  Maxwell field the form (13) we will call *outgoing one-way null field* (outgoing OWN).
- Maxwell tensor $F_{ab} = 2\varphi_2 l_{[a} m_{b]} + 2\bar{\varphi}_2 l_{[a} \bar{m}_{b]}$.
- The system of Maxwell equtions for outgoing OWN field is (in Petrov type D spacetime)

$$\begin{cases} D\varphi_2 + (2\epsilon - \rho)\varphi_2 = 0, \\ \delta\varphi_2 + (2\beta - \tau)\varphi_2 = -j_2; \end{cases} \tag{14}$$

$j_2 = j_{AA'}\iota^A\iota^{A'}$, $j_{AA'}$ — 4-current spinor.

General solution for OWN field in the Kerr spacetime

- The system for outgoing OWN field (14) in Boyer-Lindquist coordinates is ($\varphi_2 = \varphi_2(t, r, \theta, \phi)$)

$$\begin{cases} \frac{r^2+a^2}{\Delta}\frac{\partial\varphi_2}{\partial t} + \frac{\partial\varphi_2}{\partial r} + \frac{a}{\Delta}\frac{\partial\varphi_2}{\partial\phi} + \frac{1}{r-ia\cos\theta}\varphi_2 = 0, \\ ia\sin\theta\frac{\partial\varphi_2}{\partial t} + \frac{\partial\varphi_2}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial\varphi_2}{\partial\phi} + \left(\mathrm{ctg}\theta + \frac{ia\sin\theta}{r-ia\cos\theta}\right)\varphi_2 = -\sqrt{2}(r+ia\cos\theta)j_2. \end{cases} \tag{15}$$

- Solution of the system (15) is

$$\varphi_2 = \frac{1}{\sin\theta(r-ia\cos\theta)}\left(e^{F(\psi_1,\psi_2)} - \sqrt{2}\int \Sigma j_2 \sin\theta d\theta\right), \tag{16}$$

$F(\psi_1, \psi_2)$ is arbitrary function of integrals of the system (15)

$$\psi_1 = t - r - M\ln\Delta - \frac{M^2}{\sqrt{M^2-a^2}}\ln\left(\frac{r-r_+}{r-r_-}\right) + ia\cos\theta, \tag{17}$$

$$\psi_2 = \phi - \frac{a}{2\sqrt{M^2-a^2}}\ln\left(\frac{r-r_+}{r-r_-}\right) - i\ln\frac{1-\cos\theta}{\sin\theta}. \tag{18}$$

General solution for algebraically special fields

- *V. O. Pelykh, Y. V. Taistra* Class of general solutions of Maxwell equations in Kerr space-time, Mathematical methods and physicomechanical fields **59**, 1, 2016.
- *J. Jezierski, T. Smolka* A geometric description of Maxwell field in a Kerr spacetime Class. Quant. Grav. **33**, 2016.
- *P. P. Fiziev* Classes of exact solutions to the TME, Class. Quantum Grav. **27** 2010.
- *S. Chandrasekhar* On Algebraically Special Perturbations of Black Holes, Proc. R. Soc. Lond. A **392**, 1984.
- *G. F. Torres del Castillo* 3-D Spinors, Spin-Weighted Functions and their Applications, Pr. in Math. Ph. **32**, 2003.

Separation of variables for OWN field

- Let us consider the homogeneous system (15) and let us rewrite it for function $\psi(t,r,\theta,\phi) = \sin\theta(r - ia\cos\theta)\varphi_2(t,r,\theta,\phi)$[1]

$$\begin{cases} \dfrac{r^2+a^2}{\Delta}\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial r} + \dfrac{a}{\Delta}\dfrac{\partial\psi}{\partial\phi} = 0, \\ ia\sin\theta\dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial\theta} + \dfrac{i}{\sin\theta}\dfrac{\partial\psi}{\partial\phi} = 0. \end{cases} \tag{19}$$

- We are finding the solution of (19) in the form $\psi(t,r,\theta,\phi) = e^{i\omega t + im\phi}R(r)S(\theta)$. The following system of ODE's is obtained

$$\begin{cases} R'(r) + \left(i\omega\dfrac{r^2+a^2}{r^2-2Mr+a^2} + im\dfrac{a}{r^2-2Mr+a^2}\right)R(r) = 0, \\ S'(\theta) + \left(\text{ctg}\theta - a\omega\sin\theta - \dfrac{m}{\sin\theta}\right)S(\theta) = 0. \end{cases} \tag{20}$$

[1] V. O. Pelykh, Y. V. Taistra Acta Ph. Pol. B Proc. Suppl. **10**, 2, 2017.

- Solution with separable variables is

$$\varphi_2(t,r,\theta,\phi)=\frac{C}{\sin\theta(r-ia\cos\theta)}e^{-a\omega\cos\theta}\frac{(1-\cos\theta)^m}{\sin^m\theta}\times$$
$$\times\, e^{i\omega\left(t-r-M\ln\Delta-\frac{M^2}{\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|\right)+im\left(\phi-\frac{a}{2\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right|\right)}; \qquad (21)$$

$\omega\in\mathbb{R},\ m\in\mathbb{Z}$.

- It is outgoing wave solution with singularities on the axis of rotation $(\theta=0,\ \theta=\pi)$.

- The phase of wave solution (21) is (red is Schwarzshcild logarithmic phase term, blue - Kerr phase term)

$$\tilde{\phi} = m\phi + \omega\left(t - r - 2M\ln\left|r - 2M + \frac{a^2}{r}\right| + \right.$$

$$\left. + M\ln\left|1 - \frac{2M}{r} + \frac{a^2}{r^2}\right| - \frac{2\omega M^2 + a}{2\omega\sqrt{M^2 - a^2}}\ln\left|1 - \frac{2\sqrt{M^2 - a^2}}{r - M + \sqrt{M^2 - a^2}}\right|\right). \quad (22)$$

- A critical point that is found as solution of equation $\partial\tilde{\phi}/\partial r = 0$ is

$$r_{cr.} = \sqrt{-\frac{am}{\omega} - a^2}. \quad (23)$$

- If we require that such point lies out of the horizon

$$r_{cr.} > r_+ \quad (24)$$

then the wave is outgoing in the region $(r_{cr.}, \infty)$ and ingoing in the region $(r_+, r_{cr.})$. Condition (24) coincides with superradiant condition[2].

[2] Teukolsky 1973, Starobinskii, Churilov 1973.

Polarization

- To obtain electric vector let us go to locally orthogonal frame

$$\varphi_2 = -E_{\hat{1}} - iE_{\hat{2}}, \tag{25}$$

electric vector components $E_{\hat{1}}$ and $E_{\hat{2}}$ are

$$E_{\hat{1}} = \sqrt{c_1^2 + c_2^2} A \sqrt{\Sigma} \sin\left(\tilde{\phi} - \operatorname{arctg} \frac{c_1 r - c_2 a \cos\theta}{c_2 r + c_1 a \cos\theta}\right), \tag{26}$$

$$E_{\hat{2}} = \sqrt{c_1^2 + c_2^2} A \sqrt{\Sigma} \sin\left(\tilde{\phi} - \operatorname{arctg} \frac{c_1 r - c_2 a \cos\theta}{c_2 r + c_1 a \cos\theta} + \frac{\pi}{2}\right), \tag{27}$$

$$A = \frac{(1 - \cos\theta)^m \, e^{-a\omega \cos\theta}}{\sin^{m+1}\theta (r^2 + a^2 \cos^2\theta)}.$$

- The square of electric vector is

$$E_{\hat{1}}^2 + E_{\hat{2}}^2 = |\varphi_2|^2 = (c_1^2 + c_2^2) \frac{e^{-2a\omega\cos\theta}}{\sin^2\theta (r^2 + a^2\cos^2\theta)} \left(\frac{1 - \cos\theta}{\sin\theta}\right)^{2m}. \tag{28}$$

- $\omega > 0$ — right circular pol. (RCP), $\omega < 0$ — left circular pol. (LCP).
- Amplitude of RCP wave

$$A_R \sim e^{-a\omega_R \cos\theta} \qquad (29)$$

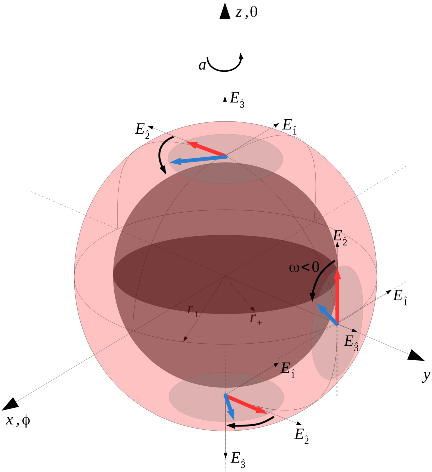
and LCP

$$A_L \sim e^{a\omega_L \cos\theta}, \qquad (30)$$

$\omega_R, \omega_L \in \mathbb{R}_+$.

- We expect the effect of suppresion of RCP in northern hemisphere ($\theta = 0$) and LCP in southern hemisphere ($\theta = \pi$).

$$a\tilde{\omega} = 2.992 \cdot 10^{24}. \qquad (31)$$



Exact polarization formula in wave optics

- Rotation of polarization angle

$$\Phi = \arctan\left(\frac{A_R^2 c_1 + A_L^2 d_1 + \frac{a\cos\theta}{r}(A_R^2 c_2 - A_L^2 d_2)}{A_R^2 c_2 + A_L^2 d_2 + \frac{a\cos\theta}{r}(A_R^2 c_1 - A_L^2 d_1)}\right), \tag{32}$$

$$A_{L,R} = \frac{1}{\sin\theta}\left(\frac{1-\cos\theta}{\sin\theta}\right)^m \frac{e^{\mp\omega a\cos\theta}}{r^2 + a^2\cos^2\theta} \tag{33}$$

- In high frequency limit and $c_1 = 0$

$$\Phi = -\arctan\left(\frac{a\cos\theta}{r}\right). \tag{34}$$

- G. Skrotskii (1957): $\Delta\Phi = 3\frac{aM}{p^2}$,

  N. Balasz (1958), J. Plebanski (1960),

  Pinault, Roeder (1977):$\Delta\Phi = 2\frac{aM}{p^2}$,

  I. Dymnikova (1987): $\Delta\Phi = 2\arctan\left(\frac{aM}{R}\right) cos\theta = 2\frac{aM}{p^2}$,

  M. Abyshev (2008): $\Delta\Phi = 3\frac{aM}{p^2}$ або $\Delta\Phi = 4\frac{3aM}{p^2}$,

- The exact analysis to the polarization of the X-rays from type AGN and BH becomes possible thanks to XTP and eXTP missions.

Summary

- The problem of electromagnetic perturbation in the Kerr spacetime still remains complicated and usually is solved by numerical and approximate methods.
- Consideration of OWN field gives partial solution in analytic form

$$\varphi_2(t,r,\theta,\phi) = \frac{C}{\sin\theta(r - ia\cos\theta)} e^{-a\omega\cos\theta} \frac{(1-\cos\theta)^m}{\sin^m\theta} \times$$
$$\times\; e^{i\omega\left(t-r-M\ln\Delta-\frac{M^2}{\sqrt{M^2-a^2}}\ln\left(\frac{r-r_+}{r-r_-}\right)\right)+im\left(\phi-\frac{a}{2\sqrt{M^2-a^2}}\ln\left(\frac{r-r_+}{r-r_-}\right)\right)}. \quad (35)$$

- Solution has coordinate dependent singularities at $\theta = 0, \pi$. We expect to find physical effects of Maxwell and Kerr field interaction beyond these these points.
- Solution describes free electormagnetic waves with right $(\omega > 0)$ and left $(\omega < 0)$ circular polarization.
- Outgoing Maxwell field with its own angular momentum interacts with Kerr black hole angular momentum. When $r_{cr.} > r_+$ holds (superradiant condition) such field is ingoing in $(r_+, r_{cr.})$ and outgoing in $(r_{cr.}, \infty)$.