Non-Schwarzschild black-hole metrics in alternative theories of gravity: analytical approximations

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based on:
K.D. Kokkotas, R.K., A. Zhidenko Phys.Rev. D96 (2017) no.6, 064004;
Phys.Rev. D96 (2017) no.6, 064007.

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SLEZSKÁ UNIVERZITA V OPAVĚ

- Weak gravity is experimentally tested and well described in term of the post-Newtonian approximations
- The era of testing strong gravity has begun only now: the current uncertainty in determination of the angular momentum and mass of the observed black holes does not allow us to discard or strongly constrain alternative theories of gravity (R. K., A. Zhidenko Phys.Lett. B756 (2016) 350; N. Yunes, K. Yagi, F. Pretorius PRD 94 (2016) no.8, 084002). Observations in the electromagnetic spectrum are promising.
- Alternative theories (emanating from the necessity to solve a number of fundamental problems, such as the nature of the spacetime singularity, approach to quantization of gravity, dark energy/dark matter problem, hierarchy problem etc.) could potentially be either discarded or constrained via consideration of observable phenomena in electromagnetic and gravitational spectra in the vicinity of black holes.
- In order to test strong gravity through black holes, we need to know the black hole solution (either numerical or exact) in a theory under consideration.
- An exact solution is much better, because it allows one to solve a broader class of problems and in much more economic way, but usually it is not possible to find an exact solution. What we suggest instead is an approximate solution.

Approximate solution must:

- be reasonably accurate and based on the convergent procedure, so that the accuracy would be controllable.
- be reasonably compact
- describe a black hole space-time well not only in some region (for example, far from black hole or, oppositely, near its horizon), but in the whole space outside the black hole.

We shall consider **two** alternative theories of gravity with higher curvature corrections for which only numerical black hole solutions are known and construct analytical approximations for these solutions:

- $D = 4$
- higher curvature corrections: the two cases we shall consider are
  higher derivative gravity (Einstein-Weyl)
  and
  Einstein-dilaton-Gauss-Bonnet gravity
- black holes are asymptotically flat and have the same post-Newtonian behavior as the Schwarzschild solution

The Einstein gravity with added quadratic in curvature term has the general form

$$I = \int d^4x\sqrt{-g}\left(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2\right) , \quad (1)$$

H. Lü, A. Perkins, C. Pope, K. Stelle [Phys.Rev.Lett. 114 (2015), 171601] For a static black hole solution, we can take $\gamma = 1$ and $\beta = 0$ without loss of generality.

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \quad (2)$$

$$p = \frac{r_0}{\sqrt{2\alpha}}. \quad (3)$$

Notice that for all p the Schwarzschild metric is the exact solution of the Einstein-Weyl equations as well, but at some minimal nonzero p_{min}, in addition to the Schwarzschild solution, there appears the non-Schwarzschild branch, which describes the asymptotically flat black hole, whose mass is decreasing, when p grows, and vanishing at some p_{max}. The approximate maximal and minimal values of p are:

$$p_{min} \approx 1054/1203 \approx 0.876, \quad p_{max} \approx 1.14 \quad (4)$$

Following the parametrization of [L. Rezzolla, A. Zhidenko, Phys.Rev. D90 (2014) no.8, 084009] we define the functions A and B through the following relations:

$$h(r) \equiv xA(x) \qquad \frac{h(r)}{f(r)} \equiv B(x)^2 \,, \tag{5}$$

where x denotes the dimensionless compact coordinate

$$x \equiv 1 - \frac{r_0}{r} \,. \tag{6}$$

We represent the above two functions as follows:

$$\begin{aligned} A(x) &= 1 - \epsilon(1-x) + (a_0 - \epsilon)(1-x)^2 + \tilde{A}(x)(1-x)^3, \\ B(x) &= 1 + b_0(1-x) + \tilde{B}(x)(1-x)^2, \end{aligned} \tag{7}$$

where $\tilde{A}(x)$ and $\tilde{B}(x)$ are introduced in terms of the continued fractions, in order to describe the metric near the event horizon $x = 0$:

$$\tilde{A}(x) = \cfrac{a_1}{1 + \cfrac{a_2 x}{1 + \cfrac{a_3 x}{1 + \cfrac{a_4 x}{1 + \ldots}}}} \,, \quad \tilde{B}(x) = \cfrac{b_1}{1 + \cfrac{b_2 x}{1 + \cfrac{b_3 x}{1 + \cfrac{b_4 x}{1 + \ldots}}}} \,.$$

At the event horizon one has: $\tilde{A}(0) = a_1, \quad \tilde{B}(0) = b_1$.

We notice that $a_0 = b_0 = 0$, i.e. the post-Newtonian parameters for the non-Schwarzschild solution coincide with those in General Relativity. We fix the asymptotic parameter ϵ as

$$\epsilon = -\left(1 - \frac{2M}{r_0}\right), \tag{8}$$

using the value of the asymptotic mass which can be found by numerical fitting of the asymptotical behavior of the metric functions.

Expanding functions h and f near the event horizon we find the parameters $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ for each value of the parameter p. The fitting of numerical data for various values of p and r_0 shows that ϵ, a_1, and b_1 can be approximated within the maximal error $\lesssim 0.1\%$ by parabolas as follows:

$$\epsilon \approx (1054 - 1203p)\left(\frac{3}{1271} + \frac{p}{1529}\right), \tag{9}$$

$$a_1 \approx (1054 - 1203p)\left(\frac{7}{1746} - \frac{5p}{2421}\right), \tag{10}$$

$$b_1 \approx (1054 - 1203p)\left(\frac{p}{1465} - \frac{2}{1585}\right). \tag{11}$$

With the same accuracy we are able to find a_2 and b_2 as

$$a_2 \approx \frac{6p^2}{17} + \frac{5p}{6} - \frac{131}{102}, \quad (12)$$

$$b_2 \approx \frac{81p^2}{242} - \frac{109p}{118} - \frac{16}{89}. \quad (13)$$

a_3 and a_4 diverge at

$$p \approx \frac{237}{223}.$$

Therefore we find out that these parameters can be well approximated as

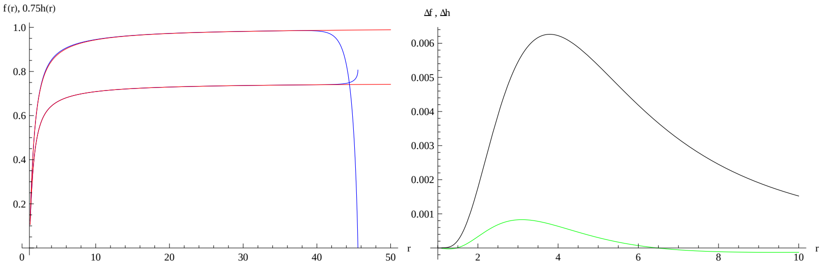
$$a_3 \approx \frac{\frac{9921p^2}{31} - 385p + \frac{4857}{29}}{237 - 223p}, \quad (14)$$

$$a_4 \approx \frac{\frac{9p^2}{14} + \frac{3149p}{42} - \frac{2803}{14}}{237 - 223p}. \quad (15)$$

In this way, although each of the parameters a_3 and a_4 diverges at $p \approx 237/223$, its ratio is finite and thereby has finite contribution into the continued fraction. Finally, we observe that b_3 and b_4 are well approximated by the straight lines as

$$b_3 \approx -\frac{2p}{57} + \frac{29}{56}, \quad (16)$$

$$b_4 \approx \frac{13p}{95} - \frac{121}{98}. \quad (17)$$



Comparison of numerical and analytical approximations for the metric functions: $r_0 = 1$, $\alpha = 0.5$ ($p = 1$). Left panel: $f(r)$ (upper) and, rescaled, $h(r)$ (lower). Numerical approximation (blue) fails at sufficiently large distance while our analytical approximation (red) has the correct behavior both near and far from the event horizon. Right panel: the difference between analytical and numerical approximations for $f(r)$ (black, upper) and $h(r)$ (green, lower). The largest difference is around the innermost stable circular orbit of a massive particle and photon circular orbit, where it still remains smaller than 0.1%.

In case one is interested in a less accurate, but more compact expression for the metric, one can be limited by the second order in the continued fraction expansion, i. e. take $a_3 = b_3 = 0$. Then, it is sufficient to consider a linear fit for ϵ, a_1, a_2, b_1, b_2. This way we obtain even simpler form for the metric functions $A(r)$ and $B(r)$

$$
\begin{aligned}
A(r) &= 1 - \frac{(1054 - 1203p)r_0^2}{2r^2} \times \left(\frac{r + r_0}{163 r_0} + \frac{11 r_0}{278(7r - 18 r_0 - 17p(r - r_0))}\right), \\
B(r) &= 1 - \frac{4(1054 - 1203p)r_0^2}{1881 r^2 (2(r + r_0) - p(r - r_0))}.
\end{aligned}
$$

The Lagrangian for dilaton gravity with a Gauss Bonnet term reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}R - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi \\ &+ \frac{\alpha'}{8g^2}e^\phi\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right),\end{aligned} \tag{18}$$

where α' is the Regge slope and g is the gauge coupling constant.
A spherically symmetric spacetime maybe chosen

$$ds^2 = -e^{\Gamma(r)}dt^2 + e^{\Lambda(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{19}$$

The numerical black hole solution for this case was found in P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, Phys. Rev. D **54**, 5049 (1996).

$$p = \frac{6\alpha'^2}{g^4 r_0^4}e^{2(\phi_0 - \phi_\infty)}, \qquad 0 \le p < 1, \tag{20}$$

and $p = 0$ corresponds to the Schwarzschild black hole.

$$e^{\Gamma(r)} \approx \left(1 - \frac{r_0}{r}\right) \frac{\mathcal{N}_1}{\mathcal{D}_1} \quad \text{and} \quad e^{(\Gamma(r)+\Lambda(r))/2} \approx \frac{\mathcal{N}_2}{\mathcal{D}_2}\,, \tag{21}$$

where

$$\begin{aligned}
\mathcal{N}_1 &= 30888 r r_0 (r + r_0)(927 r - 1060 r_0) p^6 - 3 r_0 (145693952 r^3 - 24067680 r^2 r_0 - 156948260 r r_0^2 - 5338905 r_0^3) p^5 \\
&+ (3750946056 r^4 - 3062334104 r^3 r_0 - 325162656 r^2 r_0^2 - 1478746401 r r_0^3 - 53126788 r_0^4) p^4 \\
&- 2(6293682780 r^4 - 7334803204 r^3 r_0 - 306613944 r^2 r_0^2 - 934415049 r r_0^3 + 61245382 r_0^4) p^3 \\
&+ 8(1350407212 r^4 - 2160940683 r^3 r_0 - 64904931 r^2 r_0^2 - 139116640 r r_0^3 + 62251200 r_0^4)) p^2 \\
&+ 1048(1846581 r^4 + 3798205 r^3 r_0 + 155610 r^2 r_0^2 + 270655 r r_0^3 - 321860 r_0^4) p - 7666120 r^3 (509 r - 275 r_0)\,, \\
\mathcal{D}_1 &= 11528(1 - p)(5 - 3p) r^3 \left[117(927 r - 1060 r_0) p^2 - (74741 r - 121424 r_0) p - 67697 r + 36575 r_0\right]\,, \\
\mathcal{N}_2 &= 133380 r_0^2 p^4 - 7695(58 r^2 + 38 r r_0 + 75 r_0^2) p^3 + 10(471735 r^2 - 198819 r r_0 + 78964 r_0^2) p^2 \\
&- 4(2398707 r^2 - 1567647 r r_0 + 86450 r_0^2) p + 26676 r (201 r - 151 r_0)\,, \\
\mathcal{D}_2 &= 18(13 - 9p) r \left[95(29 r + 19 r_0) p^2 - 60(419 r - 248 r_0) p + 6(3819 r - 2869 r_0)\right]\,.
\end{aligned}$$

Final remarks

- The analytical, approximate, but quickly convergent expressions for the black hole metrics are obtained for the two numerical black hole solutions in the a) quadratic (Einstein-Weyl) gravity b) Einstein-dilaton-Gauss-Bonnet gravity.
- For a number of purposes (accretion, particle motion, black hole shadows, quasinormal modes, themordyanmics, Hawking radiation etc.), these analytical expressions can serve in the same way as exact solutions.
- Rotating analogues of these solutions either even unknown (Einstein-Weyl) or known only numerically (EdGB)
- Analytical metrics for rotating black holes are coming soon...see arxiv.org.

One of the coupling constants can be fixed when choosing the system of units, so we take $\gamma = 1$. Then, the equations of motion take the form

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &- 4\alpha B_{\mu\nu} + 2\beta R (R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}) \\ &+ 2\beta (g_{\mu\nu} \Box R - \nabla_{\mu} \nabla_{\nu} R) = 0 \,, \end{aligned} \qquad (22)$$

where

$$B_{\mu\nu} = (\nabla^{\rho} \nabla^{\sigma} + \frac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma} \qquad (23)$$

is the tracefree Bach tensor. It is the only conformally invariant tensor that is algebraically independent of the Weyl tensor.

One can write a static metric as follows

$$ds_4^2 = -\lambda^2 \, dt^2 + h_{ij} \, dx^i dx^j \,, \tag{24}$$

where λ and h_{ij} are functions of the spatial coordinates x^i. In H. Lü, A. Perkins, C. Pope, K. Stelle [Phys.Rev.Lett. 114 (2015), 171601] it was shown that, taking the trace of the field equations (22) and integrating the equations of motion over the spatial domain from the event horizon to infinity, one can find that

$$\int \sqrt{h} \, d^3x \Big[D^i(\lambda R D_i R) - \lambda (D_i R)^2 - m_0^2 \lambda R^2 \Big] = 0 \,, \tag{25}$$

where D_i is the covariant derivative with respect to the spatial 3-metric h_{ij}.
By definition, λ vanishes on the event horizon, so that if $D_i R$ goes to zero sufficiently rapidly at spatial infinity, then the total derivative term can be discarded and any static black-hole solution of (1) must have vanishing Ricci scalar $R = 0$. The latter means that, without loss of generality, we can be constrained by the Einstein-Weyl gravity ($\beta = 0$). Then, since $B_{\mu\nu}$ is tracefree, the trace of (22) implies the vanishing Ricci scalar ($R = 0$). Therefore, the Schwarzschild solution is also a solution for the Einstein-Weyl gravity.
Summarizing, when considering static solutions in the most general Einstein gravity with quadratic in curvature corrections given by (1), one can take $\gamma = 1$ and $\beta = 0$ without loss of generality.