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QUANTUM GRAVITY AND NONCOMMUTATIVE SPACE-TIMES

1. **Field - theoretic path: from Maxwell electrodynamics to basic model of fundamental interactions (Standard Model + gravity)**
2. **Conceptual problem with QFT approach to QG: dynamical nature of quantum space-times and their NC structures**
3. **Three most popular models of NC space-times (Snyder, κ-Minkowski, $\theta_{\mu\nu}$-deformed (DFR))**
4. **On the measurability of QG effects**
5. **Conclusions**

QG ≡ quantum gravity
NC ≡ noncommutative
DFR ≡ Doplicher, Fredenhagen, Roberts

1. FIELD-THEORETIC PATH: FROM MAXWELL ELECTRODYNAMICS TO BASIC MODEL OF FUNDAMENTAL INTERACTIONS

1865 - Maxwell' electrodynamics: classical relativistic field theory - first building block of fundamental interactions framework.

1905 - relativistic covariance of electrodynamics becomes explicite in four-tensor notation (potential $A_\mu(x) \Rightarrow$ field strength $F_{\mu\nu}(x)$).

First local gauge theory: the choice of electromagnetic dynamics linked with local gauge symmetry!

1915 - Einstein gravity theory - historically second part of the framework of fundamental interactions (prepotential $g_{\mu\nu}(x) \rightarrow$ potential (connection) $\Gamma_{\mu\nu}^{\ \ \rho}(x) \rightarrow$ curvature $R_{\mu\nu}^{\ \ \rho\tau}(x)$).
gravity $\leftrightarrow$ gauge theory with constraints

Einstein - Hilbert action:

$$S_{EH}^{\text{grav}} = -\frac{1}{2\kappa}\int d^4x\sqrt{\det g_{\mu\nu}}\,R \qquad R = R_{\mu\nu}^{\ \ \mu\nu}$$

(scalar curvature)

Covariance under local space-time transformations (diffeomorphisms)

$$x'_\mu = x_\mu + \xi_\mu(x) \qquad \xi_\mu(x) - \text{arbitrary functions}$$

Einstein equations for gravitational fields:

$$G_{\mu\nu}(x) = R_{\mu\nu}(x) - \tfrac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x) \qquad \kappa = \frac{8\pi G}{c^4} \leftarrow \text{Newton constant}$$

$\uparrow$ Ricci tensor $R_{\mu\nu} = R^{\tau}_{\mu\tau\nu}$

$\uparrow$ local energy-momentum tensor

Special feature of gravitational field: it describes dynamically the geometry of physically possible curved space-times

invariant length element $\Longrightarrow$ $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

$\Uparrow$ local metric

gravity $\Longleftrightarrow$ dynamical theory of curved space-time

Matter described by $T_{\mu\nu}(x)$ determines the space-time curvature $R^{\mu\nu}{}_{\rho\tau}$

Basic question after 1915: how to describe the matter (rhs of Einstein equations) on fundamental level?

Answer: by classical and subsequently quantized field theory.

Relativistic fundamental free fields are characterized by mass m and spin $s = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ (Wigner 1939), namely

1925 - scalar Klein-Gordon field ($m > 0, s = 0$) (e.g. Higgs particle)

1927 - spinorial Dirac field ($m > 0, s = \frac{1}{2}$) (e.g. electron/positon)

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Next important step: gauge fields with nonAbelian internal symmetries

1954 - Yang-Mills (YM) fields ($A_\mu \to A_\mu^{ij}$, $F_{\mu\nu} \to F_{\mu\nu}^{ij}$)

electrodynamics		chromodynamics
$(A_\mu, \Psi_A \quad \mathrm{A}{=}1 \ldots 4)$	$\longrightarrow$	$(A_\mu^{ij}, \Psi_A^{i;\alpha} \quad (i, j = 1, 2, 3))$
photons + electrons		gluons+quarks

~ 1970 - field-theoretic Standard Model (SM).

All fundamental interactions described by two sectors:

- gravitational ($g_{\mu\nu}$ or e^a_μ, where $g_{\mu\nu} = e^a_\mu e_{\nu a}$)
- SM of elementary particles (QCD + electroweak sector - fields Ψ)

Standard tool of calculations: quantum local relativistic field theory in perturbative approach. Full action (postulated as valid for all energies)

$$S^{\mathrm{TOT}} = \underset{\substack{\uparrow \\ \text{canonically quantized}}}{S^{\mathrm{grav}}}[g_{\mu\nu}] + S^{\mathrm{SM}}[\Psi] + \underset{\substack{\uparrow \\ \text{weak gravitational perturbation}}}{S^{\mathrm{int}}}[g_{\mu\nu}, \Psi]$$

In perturbative calculations (Feynman diagrams) do appear infinities, which however may be removed by renormalization procedure.

Remark: Quantized Einstein gravity can be introduced as suitably constrained local Poincaré gauge QFT of massless spin two metric field with local gauge parameter $\xi_\mu(x)$ (Kibble 1960, Ogievetsky etc. 1965).

Important: if we employ Higgs mechanism the field-theoretic SM sector is renormalizable, but unfortunately

Einstein QG is not renormalizable in perturbative approach!

2. CONCEPTUAL PROBLEM WITH QFT APPROACH TO QG: DYNAMICAL NATURE OF QUANTUM SPACE-TIMES AND THEIR NC STRUCTURES

Two sources of problems with field-theoretic description of QG:

i) **Technical: Nonlinearity of Einstein-Hilbert action and the dimensionfull nature of coupling constant $\kappa \sim G$ lead to D=4 perturbative nonrenormalizability (in D=4 only simple polynomial actions are renormalizable).**

ii) **Conceptual: QG describes the quantized geometrodynamics of space-time, what is not incorporated into the framework of standard field theory usually defined on static (usually flat) space-time (e.g. QED fields).**

quantum field: $$\widehat{\phi}(x) \sim \sum \widehat{a}(\vec{p})\, e^{ipx}$$

↑ quantized field oscillators

← parameter x_μ due to QG should be also quantized!

Classical space-time $x_\mu \overset{QG}{\Rightarrow}$ quantum space-time $\widehat{x}_\mu$.

In quantum model of fundamental interactions (gravity + elementary particles) due to the presence of gravity one should treat space-time as a dynamical quantum object, in particular

standard QM		QM in presence of QG
standard QFT	$\overset{QG}{\Rightarrow}$	QFT in presence of QG
$[x_\mu, x_\nu] = 0$		$[\hat{x}_\mu, \hat{x}_\nu] \neq 0$

Deduction in QM of Heisenberg algebra from uncertainty relations

microscope thought experiment:	$\Delta x_i \, \Delta p_i \geqslant \hbar$	$\longleftrightarrow$	$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$	Heisenberg algebra

In standard QM uncertainties $\Delta x_i, \Delta x_j$ and $\Delta p_i, \Delta p_j$ $i \neq j$ can be arbitrarily small

(a)	$\Delta x_i \, \Delta x_j \geqslant 0$	$\longleftrightarrow$	$[x_i, x_j] = 0$	$\Leftarrow$	classical three-space
(b)	$\Delta p_i \, \Delta p_j \geqslant 0$	$\longleftrightarrow$	$[p_i, p_j] = 0$	$\Leftarrow$	classical three-momentum

However in QM the presence of magnetic field $\vec{H}$ can dynamically modify the relations (b): $[\hat{p}_i, \hat{p}_j] = i\, e\, \varepsilon_{ijk} \hat{H}_k$. What modifies (a)?

Answer: the presence of dynamical gravitational field - modification of (a)

It has been shown that one can not measure two different space coordinates with arbitrary accuracy - Einstein eq. + Heisenberg uncertainty principle imply for space coordinates (Mead 1964)

$$(i \neq j = 1,2,3) \qquad \Delta x_i \Delta x_j \geqslant l_p^2 \qquad l_p^2 = \frac{hG}{c^3} \simeq 10^{-33}\text{cm}$$

G - Newton Constant
l_p – Planck length

These relations have been extended to relativistic form of space-time uncertainty relations (Doplicher, Fredenhagen, Roberts 1994–95)

$$\sum_{1 \leqslant j < k \leqslant 3} \Delta x_j \Delta x_k \geqslant l_p^2 \qquad \Delta x_0 \sum_{j=1}^{j=3} \Delta x_j \geqslant l_p^2$$

From algebraic formulation one gets the same uncertainty relations if

$$[\hat{x}_\mu, \hat{x}_\nu] = i\, l_p^2 \theta^{(0)}_{\mu\nu} + \ldots = l_p^2 f_{\mu\nu}\left(\frac{\hat{x}}{l_p}\right) \qquad \left(\Rightarrow f_{\mu\nu}\left(\frac{\hat{x}}{l_p}, l_p \hat{p}\right)\right) \qquad \theta^{(0)}_{\mu\nu} = -\theta^{(0)}_{\nu\mu}$$

$\uparrow\uparrow$ expanded in powers of l_p $\qquad$ $\uparrow\uparrow$ example: Snyder space-time

Various models of quantum space-time ⇒ different choices of $f_{\mu\nu}$.
Bohr: limits of measurability determine the kinematical QM algebra.

Physically noncommutative structure of quantum space-time is linked with gravitational creation mechanism of microscopic black holes

restrictions of the localization measurement in quantum space-time	$\Leftrightarrow$	creation of mini-black holes by increasing the energy density in measurement

Old idea - first Michael Bronstein (1935):

"We can not localize arbitrarily large mass (energy) in very small volume - we shall be not able to observe it because the presence of gravitational forces will make the measurement impossible"

In consequence the gravitational forces during localization measurement lead to the effective atomization of quantum space-time

reaction of dynamical space-time to quantum measurement process	$\Leftrightarrow$	below 10^{-33} cm the notion of classical space-time looses operational meaning

In QG (as earlier in QM) the noncommutativity derived from measurement restrictions becomes real structural property of the theory!

The influence of QG effects on geometric space-time structures:

classical geometry	$\overset{QG}{\Rightarrow}$	quantum geometry
$\Updownarrow$		$\Updownarrow$
continuous space-time		noncommutative space-time $(\sim l_p^2)$
$\Updownarrow$		$\Updownarrow$
points		cells of space-time with volume $\sim (l_p)^4$ in D=4 - Planck scale discreteness

Three phase space structures for three basic dynamical frameworks:

Classical mechanics and FT $\hbar = 0, G = 0$ x_μ, p_μ commutative	$\longrightarrow$	QM and QFT without QG $\hbar \neq 0, G = 0$ noncommutative phase space (x_μ commutative)	$\longrightarrow$	QM and QFT in presence of QG $\hbar \neq 0, G \neq 0$ noncommutative space-time $\hat{x}_\mu$ (quantum geometries)

Quantum geometries are related with new mathematics - quantum groups, quantum spaces, noncommutative differential geometries etc., developed in eighties and nineties: (Faddeev, Woronowicz, Drinfeld, Connes, Majid, ...)

- quantum symmetries $\longleftrightarrow$ quantum groups described by Hopf algebras which contain algebraic sector (algebra A) and coalgebraic sector (coproducts $\Delta: \; A \to A \otimes A$).
  Coproducts describe the realization of algebra A on $A \otimes \cdots \otimes A$
  deformed QM (Hilbert space) $\overset{\Delta}{\to}$ deformed QFT (Fock space)
- quantum spaces described by an algebra X, usually introduced as modules (NC representations) of Hopf algebras
- recently important extensions of Hopf algebras:
  Hopf algebras $\longrightarrow$ Hopf algebroids
  Hopf algebroids are Hopf algebras over noncommutative ring B

$$x \otimes y \quad \longrightarrow \quad x \underset{B}{\otimes} y \qquad \text{(Takeuchi 1977)}$$

  Physical application: quantum-deformed NC phase spaces are described by Hopf algebroids (bialgebra $\longrightarrow$ bialgebroids).
  If one uses standard tensor product – nonuniqueness of coproducts of bialgebroids $\longrightarrow$ coproduct gauge

3. THREE MOST POPULAR MODELS OF NC SPACE-TIMES (SNYDER, κ-MINKOWSKI, $\theta_{\mu\nu}$-DEFORMED (DFR))

a) Snyder model (1947)

Proposed to provide regularization of infinities in QFT – before invention of renormalization procedure in 50's. Then forgotten till around 2000.

Based on Lie-algebraic structure (no quantum deformation!)

Cosmological distances de-Sitter space-time geometry		Ultrashort distances NC Snyder space-time geometry
$[\hat{p}_\mu, \hat{p}_\nu] = \frac{i}{R^2} M_{\mu\nu}$	$\xleftrightarrow[\text{duality}]{\text{Born}}$ $x \Leftrightarrow p$	$[\hat{x}_\mu, \hat{x}_\nu] = i\, l_p^2 M_{\mu\nu}$
$R \simeq 10^{29}$cm - radius of the Universe noncommutative $\hat{p}_\mu$	$\xleftrightarrow[\text{duality}]{\text{macro–micro}}$	$l_p \simeq 10^{-33}$cm - Planck length noncommutative $\hat{x}_\mu$

Planckian fundamental units (Planck 1899!)

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 1.6 \cdot 10^{-33}\text{cm} \quad m_p = \sqrt{\frac{\hbar c}{G}} \simeq 10^{-5} g \quad t_p \simeq \sqrt{\frac{G\hbar}{c^5}} \simeq 10^{-43}\text{sec}$$

$l_p = \frac{\hbar c}{m_p}$ is the Compton length for m_p $\Rightarrow$ $l_p = m_p^{-1}$ if $\hbar = c = 1$

Snyder model for spinless system:

$$M_{\mu\nu} = -i\,(\hat{x}_\mu\,\hat{p}_\nu - \hat{x}_\nu\,\hat{p}_\mu) \qquad (\text{spin part } S_{\mu\nu} = 0)$$

One gets Snyder quantum phase space:

$$[\hat{x}_\mu, \hat{x}_\nu] = -il_p^2(\hat{x}_\mu\hat{p}_\nu - \hat{p}_\mu\hat{x}_\nu)$$

Extension to whole quantum phase space (consistent with Jacobi identities)

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar(\eta_{\mu\nu} + l_p^2\,\hat{p}_\mu\hat{p}_\nu) \qquad [\hat{p}_\mu, \hat{p}_\nu] = 0$$

Advantage of Snyder model: introduced NC space-time does not break Lorentz covariance

$$[M_{\mu\nu}, \hat{x}_\rho] = i(\hat{\eta}_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu)$$

One can add consistently second parameter α ($[\alpha] = L^2$)

$$[\hat{x}_\mu, \hat{p}_\nu] = i\left[\eta_{\mu\nu}(1 + \alpha\hat{p}^2) + l_p^2\,\hat{p}_\mu\hat{p}_\nu\right]$$

$$[\hat{x}_\mu, \hat{x}_\nu] = -i(l_p^2 + 2\alpha)(1 + \alpha\hat{p}^2)M_{\mu\nu} \quad \Rightarrow \quad [\hat{x}_\mu, \hat{x}_\nu] = 0 \quad \text{iff} \quad \alpha = -\frac{l_p^2}{2}$$

i.e. one gets via parameter α the link between Snyder and classical space-time.

b) κ-Minkowski NC space-time (1991–94)

Problem ~ 90's: how to introduce the quantum generalization of Poincaré algebra which incorporates third fundamental frame - independent constant (besides c and $\hbar$): the Planck length l_p or Planck mass m_p

Answer: κ-deformed Poincaré algebra (1991 JL+Nowicki+Ruegg+Tolstoy)

κ-deformed Poincaré algebra	as	quantum symmetry algebra	$\Rightarrow$	its representation is NC κ-deformed Minkowski space-time

κ-Minkowski space-time ($c = \hbar = 1$; κ – fundamental mass)

$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i$ ($\kappa \leftrightarrow m_p$ is a physical assignement)

$[\hat{x}_i, \hat{x}_j] = 0$ $\longleftarrow$ commuting NR space, only time “quantum”

Advantage of κ-deformation: nonrelativistic physics remains not deformed – κ-deformation is an ultra relativistic modification, important for very large energies/momenta.

However: $\hat{x}_\mu$ is not a Lorentz fourvector – Lorentz invariance broken!

Subtle point: mathematically Poincaré-Hopf algebra is equivalently defined if the generators are nonlinearly transformed $(M_{\mu\nu} = (M_r, N_r))$

$$P_\mu \rightarrow \tilde{P}_\mu = \tilde{P}_\mu(P_\mu) \qquad \tilde{M}_{\mu\nu} = \tilde{M}_{\mu\nu}(M_{\mu\nu}, P_\nu) \quad \leftarrow \quad \text{change of basis}$$

Standard basis: $[M, M], [M, N]$ undeformed, $[N, N]$ deformed.
Bicrossproduct basis: Lorentz algebra undeformed, $[M_{\mu\nu}, P_\rho]$ deformed
κ-deformation defines Lie algebra with p_0-dependent structure constants.

Physical question: which basis of fourmomentum P_μ is "physical"?

Usually assumed answer: bicrossproduct basis (Majid, Ruegg 1994)

In bicrossproduct such basis Lorentz algebra is not deformed, but generators P_μ break Lorentz covariance and mass Casimir is κ-deformed:

$$C_2 = p_\mu p^\mu = p_0^2 - \vec{p}^2 = \xrightarrow[\kappa \text{ finite}]{} (2\kappa \sinh \frac{p_0}{2\kappa})^2 - e^{\frac{p_0}{\kappa}} \vec{p}^2 \quad \Leftarrow \quad \kappa\text{-deformed mass-shell}$$

Consequence: energy-dependent light velocity $c \rightarrow c(E)$.

c) $\theta_{\mu\nu}$-deformed NC space-time (DFR model) (DFR ≡ Doplicher, Fredenhagen, Roberts)

$$[\hat{x}_\mu, \hat{x}_\nu] = i l_p^2 \theta^{(0)}_{\mu\nu} \qquad \theta^{(0)}_{\mu\nu} = -\theta^{(0)}_{\nu\mu}$$ – constant normalized tensor

Such choice of NC space-time was introduced firstly as NC quantum space, just as algebra – however later it was introduced as the NC representation of quantum group given by particular Hopf algebra $\mathbb{H}_\theta$, with $\theta^{(0)}_{\mu\nu}$ describing the set of six deformation parameters breaking Lorentz invariance.

$\mathbb{H}_\theta$ belongs to simpler class of quantum groups, which provides an example of so-called twist quantization of Poincaré symmetries:

- – Poincaré algebra is not modified, remains classical
- – Classical (primitive) coproducts $\Delta_0(\hat{g}) = \hat{g} \otimes 1 + 1 \otimes \hat{g}$ ($\hat{g}$ - Poincaré generators) modified by the similarity map in terms of twist factor $F = F_{(1)} \otimes F_{(2)}$

$$\Delta_{(\theta)}(\hat{g}) = F^{-1} \circ \Delta_0 \circ F \qquad F = \exp\left[\frac{i}{2}\theta^{\mu\nu}\left(P_\mu \otimes P_\nu - P_\nu \otimes P_\mu\right)\right]$$

Twisted coproducts change the classical multiparticle sectors but the single particle states (IRRep of Poincaré algebra) remain classical.

Advantage of dealing with twist quantization: we have explicite formulae expressing NC fields $\phi(\hat{x}), \chi(\hat{x})$ by standard fields $\phi(x), \chi(x)$ - by introducing of so-called star-product ($\star$-product) multiplication:

$\phi(\hat{x}) \cdot \chi(\hat{x})$ $\hat{x}_\mu$ – κ-Minkowski noncommutative	representation $\longrightarrow$ Weyl map	$\phi(x) \star \chi(x)$ $\hat{x}_\mu$ – commutative	$\leftarrow$	classical realization of NC quantum fields algebra

where $\star$ describes nonlocal product; for the case of $\theta_{\mu\nu}$-deformation the $\star$-product has been introduced much earlier in statistical mechanics as nonlocal Moyal product:

$$\phi(x) \star \chi(x) = \phi(x) \exp\left(\frac{\overleftarrow{\partial}}{\partial x^\mu} \theta^{(0)}_{\mu\nu} \frac{\overrightarrow{\partial}}{\partial y_\nu} \right) \chi(y) \Big|_{x=y}$$

derivatives of
arbitrary order!

This is the reason that most of explicitly calculated deformations of QFT models are presented with insertion of $\theta_{\mu\nu}$ - deformed NC space-time – also because the calculations are relatively simple.

Disadvantages: $\theta^{(0)}_{\mu\nu}$ breaks space–time isotropy, and the classical QFT representing $\theta_{\mu\nu}$-deformed NC model is nonlocal.

4. ON THE MEASURABILITY OF QG EFFECTS

"Planck window" still closed – no firm experimental evidence of QG effects "QG Phenomenology" however appears as an active research field: with prediction of possible direct and indirect observable effects in QG models.

Most frequently considered fundamental effects:

- modification of light velocity c (time delay in gamma ray bursts)
- violation of Lorentz symmetry
- QG effects in description of violent cosmic collisions (black holes) (quantum deviations from classical Einstein gravity calculations)

Important problem: many estimates of QG phenomenology are calculated on the base of particular approach to QG - there are deduced the upper bounds on some QG - induced parameters (e.g. deviations from c).

Example: Domokos et all (1994) – by studying time delays in the arrival of photons from distant areas of the Universe estimated in particular frame of κ-deformed theory that $\kappa > 10^{14}\mathrm{GeV}$ (assumption $\kappa = m_p$ can be made).

There are also theoretical calculations (Aschieri 2006, Banerjee 2010 etc.) providing the QG corrections due to the deformation procedure

$$\begin{matrix}\text{classical Riemannian} \\ \text{geometry}\end{matrix} \xrightarrow[\text{deformation}]{} \begin{matrix}\text{noncommutative} \\ \text{Riemann geometry}\end{matrix}$$

For large class of deformations (using $\star$-product technique) in NC Einstein theory one gets the corrections proportional to l_p^2 – far from observability (surprising lack of correction linear in l_p, however no general proof).

The best chances to find QG effects: astrophysical measurements (e.g. Planck satelite) – ultra energetic signals from early Universe, QG corrections e.g. from inflation period are augmented by long time flow.

Resume: "Planck window" opens for QG phenomenology in astrophysics, but rather very slowly – classical relativity remains amazingly effective.

Comment: If we consider QG as a sector of quantized (super)string theory there is a second "string length" parameter l_s related with string tension ($l_s > l_p$ by few orders).

5. CONCLUSIONS

Our experimental and theoretical tools are still too coarse for detecting directly QG effects - QG phenomenology is at early stage. However
– astrophysical measurements give substantial hopes if properly linked with theoretical and numerical nonperturbative calculations (e.g. QG corrections to large black holes collisions, observed via gravitational waves etc.)
– important step in QG formalism which is lacking: NC structures studied till present are static (numerical deformation parameters!), usually originating from quantum groups, but NC factors should be dynamical, determined by still to be discovered additional QG equations, becoming trivial in classical commutative limit. See however

- Some progress done in loop quantum gravity (see JKG talk)
- If QG with all elementary interactions are described by quantum (super) string theory – recently interesting proposal (Freidel, Leigh, Minic):
  dynamical curved phase space → modular (super)strings
- AdS/CFT duality (more generally gravity/gauge dualities) – since 1998 provide still misterious link between gravity and matter sectors.

MAIN MESSAGE:

Future QG still definitely not known, but theoretical research within different complementary approaches (loop quantum gravity, lattice approaches to QG functional integral, quantum string theory, NC curved geometries etc.) as well as experimental efforts to open the Planck window are both very important and desired!

THANK YOU