Particle content of gravity theories with Lagrangians explicitly depending on the Weyl tensor

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This is an advanced version of a research programme whose preliminary version was presented at POTOR 3.

The programme:

to restrict the possible number of gravity theories.

,,Possible" = theoretically tenable.

There is INFINITY of gravity theories, all being extensions of Einstein's GR.

GR is very susceptible to various extensions going in all imaginable directions.

They are candidates for the true theory of gravitation because experimental data on gravitation are very scarce.
Theoretical motivation:
— string theory,
— quantum gravity,
— accelerated expansion of the Universe (instead of dark energy),
— easy to create.
Most investigated theories of gravity:
metric nonlinear gravity (NLG) theories —differ from GR in one axiom:
$L = R$ is replaced by $L = f(g_{\mu\nu}, R_{\alpha\beta\mu\nu})$ — any smooth scalar function.

How to choose a narrow class of tenable $L = f$?

1. Positivity of energy?

For $L = f(R) = R + aR^2 + bR^3 + \dots$

All theories with $a > 0$ have positive energy density $\Rightarrow$ one half of L is discarded $\Rightarrow$ infinity of L remain.

2. Existence of the Newtonian limit?

Each $L = f(R) = R + aR^2 + bR^3 + \dots$ for $a > 0$ admits the Newtonian limit.

No other criteria are known.

Mathematically: each $L = f(g_{\mu\nu}, R_{\alpha\beta\mu\nu})$ gives rise to a dynamically consistent theory.

First assumption:
NLG theory is a specific CFT and obeys its rules.
Axiom of CFT and QFT:
each fundamental field has a definite mass ≥ 0 **and spin**.
Fundamental field — is in Lagrangian.
$\Rightarrow$ if a field in L has NO definite mass and spin, it is a *unifying field* $\equiv$ mixture of a number of distinct fundamental fields.
NLG: the metric in $L = f(g_{\mu\nu}, R_{\alpha\beta\mu\nu})$ is just a unifying field for a *multiplet of gravit. fields*.
One field field in the multiplet: the spacetime metric —identical to $g_{\mu\nu}$ or not.
Gravity in NLG theory — NOT pure geometry (unlike in GR).

Conclusion:

non–geometric fields in gravit. multiplet must behave as known matter fields (mass and spin).

NLG: NO inherent method of decomposing the unifying $g_{\mu\nu}$ into the multiplet.

Many papers: *ad hoc* decomposition („auxiliary fields") without being fully aware of what is done.

The best known decomposition method:

generalized covariant Legendre transformations.

Close to canonical formalism in mechanics (classical and relat.), analogous to the non–covariant Hamiltonian formalism in field theory and GR.

Second assumption:
L **in NLG theory is simple**.
Motivation:
— Lagrangians of known fields (the Standard Model) are the simplest possible (Einstein),
— for complicated $f(R_{\alpha\beta\mu\nu})$ the formalism is formidably intricate.
Lagrangian of Yang–Mills fields is quadratic in ,,velocities".
Postulate:
essential features of NLG theory (without Weyl tensor) should be exhibited by a **quadratic** theory,

$$L = \kappa R + aR^2 + bR_{\mu\nu}R^{\mu\nu}, \qquad \kappa = \frac{c^3}{16\pi G}.$$

Irreducible parts of $R_{\mu\nu}$:
R and $S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$.
Two conjugate momenta:

$$\chi + \kappa \equiv \frac{\partial L}{\partial R} \quad \text{and} \quad \pi^{\mu\nu} \equiv \frac{\partial L}{\partial S_{\mu\nu}}.$$

Gravitational **triplet**: $\{g_{\mu\nu}, \chi, \pi^{\mu\nu}\}$.
Analogous decomposition into a multiplet is possible for **any** $L = f(g_{\mu\nu}, R_{\alpha\beta\mu\nu})$.

Problem:
do **all** the fields in a gravit. multiplet for any L behave as known matter fields (mass and spin)?
If NOT $\Rightarrow$ this L must be rejected on **physical grounds**.
Almost all physics (quantum!) is formulated in flat spacetime. $\Rightarrow$
Properties of fundamental fields — for **free** (non–interacting) fields.
Free fields — interactions are ,,switched off'' $\Rightarrow$ coupling constants vanish.
Simple example in flat spacetime:
charged scalar field interacting with the EM field,

$$L = -\frac{1}{2}(\partial^\mu \phi + ieA^\mu \phi)(\partial_\mu \bar{\phi} - ieA_\mu \bar{\phi}) - \frac{m^2}{2}\phi\bar{\phi} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}.$$

The field eqs.

$$\eta^{\mu\nu}\phi_{,\mu\nu} + 2ieA^{\mu}\phi_{,\mu} + ieA^{\mu}{}_{,\mu}\phi - e^2 A^{\mu}A_{\mu}\phi - m^2\phi = 0$$

and

$$\partial_{\nu}F^{\mu\nu} + 4\pi(ie\bar{\phi}\phi^{,\mu} - ie\phi\bar{\phi}^{,\mu} - 2e^2A^{\mu}\phi\bar{\phi}) \equiv \partial_{\nu}F^{\mu\nu} + j^{\mu} = 0.$$

$F_{\mu\nu}$ is free iff $\phi = 0$.
ϕ is NOT free for $F_{\mu\nu} = 0$ since $A_{\mu} = \partial_{\mu}f$, f — arbitrary $\Rightarrow$ the current $j^{\mu} \neq 0$.
$j^{\mu} = 0$ iff $e = 0 \Rightarrow \phi$ — neutral complex K–G field with mass $= m$ and spin $= 0$.

GR: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.
One recovers flat space physics for $g_{\mu\nu} = \eta_{\mu\nu}$ AND $G \to 0$.
If $G \neq 0$: $\quad G_{\mu\nu} = 0 \Rightarrow T_{\mu\nu} = 0 \Rightarrow$ no matter (positive energy density).
Non–interacting gravitationally matter $\Leftrightarrow G = 0$.
NLG theory: non–geometric fields in gravit. multiplet are specific ,,matter" fields $\Rightarrow$ each of them may independently live in Minkowski space.
NLG theory: L contains some **dimensional constants = coupling constants**.
To get the multiplet fields free $\Rightarrow$ these constants must go to 0 or ∞.

The quadratic L:

$$L = \kappa R + aR^2 + b\, R_{\mu\nu} R^{\mu\nu}.$$

Free fields are determined in eqs. of motion.
Can the momenta χ and $\pi^{\mu\nu}$ independently live in Minkowski space?

$$\chi + \kappa \equiv \frac{\partial L}{\partial R} \quad \text{and} \quad \pi^{\mu\nu} \equiv \frac{\partial L}{\partial S_{\mu\nu}}.$$

Eq. for χ

$$(3a + b)\Box\chi - \frac{1}{32\pi G}\chi = 0.$$

1. $3a + b = 0 \Rightarrow \chi = 0$ — dynamically. Remains $\pi^{\mu\nu}$ and $G \to 0$ is sufficient.
$\pi^{\mu\nu}$ obeys a nonlinear tensorial K–G eq. $\Rightarrow$ has $m_\pi^2 > 0$ and spin=2.
This is *theoretically tenable* NLG theory.

2. $3a + b > 0$.

Assumptions:

$G \to 0$, $b \to -\infty$, $a = \xi b$, $\xi = \mathrm{const} > 1/3$,
$Gb \to -\mu^2 < 0$, $\mu = \mathrm{const} > 0$.

In this limit:

$$\chi = C\pi^{\alpha\beta}{}_{;\alpha\beta}, \qquad C = \mathrm{const}.$$

χ is NOT an independent physical field, is auxiliary.

If $\chi = 0$:

$$\Box\pi_{\mu\nu} - \frac{1}{\mu^2}\pi_{\mu\nu} = 0,$$

mass$= 1/\mu$ and spin $=2$.

Conclusion:

the scalar χ is unphysical and must be excluded at the outset: already in L.
The restriction $3a + b = 0$ is *required by physics*:
χ is generated by $\pi^{\mu\nu}$ and has no its own independent existence.

The Weyl tensor contribution

The simplest Lagrangian

$$L = \kappa R + \frac{1}{3m^2}(R^2 - 3\, R_{\mu\nu} R^{\mu\nu}) + \frac{1}{k} f(W),$$

$W \equiv C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}$.
The Gauss–Bonnet topological invariant in $d = 4$:

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 = \mathrm{div}$$

$\Rightarrow f(W) = W$ may be excluded from $L \Rightarrow L$ *cannot be* purely quadratic,

$$f(W) = W^n, \quad n > 1.$$

There are two ways to define a momentum σ canonically conjugate to $C_{\alpha\beta\mu\nu}$ — a tensorial or a scalar field.
In both the cases:
after making appropriate limits in coupling constants κ, m and k, one gets in flat spacetime 2 fields, $\pi^{\mu\nu}$ and σ and
— $\pi^{\mu\nu}$ is a K–G field, mass > 0 and spin $= 2$,
— σ is NOT an independent physical field.
Final conclusion:
from the purely field–theoretical viewpoint the Weyl tensor should NOT appear in L *of any NLG theory, only* $L = f(g_{\mu\nu}, R, R_{\alpha\beta})$ *are admissible (with various restrictions)*.