# Game of frames: Jordan vs. Einstein - dynamical systems approach

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# Cosmology

General Theory of Relativity

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left(R-2\Lambda\right) + S_m$$

Friedmann-Robertson-Walker symmetry

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{r,0}\left(\frac{a}{a_0}\right)^{-4} + \dots$$

$$S_T = S - \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\nabla^\alpha\phi\,\nabla_\alpha\phi + 2U(\phi)\right)$$

- inflationary epoch – inflaton
- current accelerated expansion – quintessence

# Methodology

**Working cosmological model**:

"BB" $\rightarrow$ Inflation $\rightarrow$ RDE $\rightarrow$ MDE $\rightarrow$ de Sitter

**Dynamical system theory**:

an unstable node $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a stable node/attractor

or without BB singularity

an unstable node $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a stable node/attractor

# The theory I

"Starobinsky's" model

$$S_g = \int \mathrm{d}^4x\sqrt{-g}\Big\{-2\phi_0\Lambda + \phi_0 R + c_1 R^2\Big\}$$

introducing the scalar field

$$\varphi = \phi_0 + 2c_1 R$$

we obtain

$$S_g = \int \mathrm{d}^4x\sqrt{-g}\Big\{\varphi R - 2\phi_0\Lambda - \frac{(\varphi-\phi_0)^2}{4c_1}\Big\}$$

# The Jordan frame model

The geometry is given by the flat Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right).$$

New dimensionless variables

$$x = \frac{\dot{\varphi}}{H\varphi}, \quad z = \frac{1}{v} = \frac{\varphi}{\phi_0}.$$

The energy conservation condition is

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\Lambda,0} + \Omega_{\tilde{\Lambda},0}(z-1)^2}{z(1+x)},$$

and the acceleration equation in the following form

$$\frac{\dot{H}}{H^2} = -2 + 2\Omega_{\tilde{\Lambda},0}\frac{z(z-1)}{\Omega_{\Lambda,0} + \Omega_{\tilde{\Lambda},0}(z-1)^2}(1+x),$$

where $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$, $\Omega_{\tilde{\Lambda},0} = \frac{\tilde{\Lambda}}{3H_0^2}$ and $\tilde{\Lambda} = \frac{\phi_0}{8c_1}$ is the Einstein frame cosmological constant.

# The Jordan frame model – dynamics

The dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}\ln a} = -x(x+1) - x\left(\frac{\dot{H}}{H^2}+2\right) + 4\frac{\alpha+1-z}{\alpha+(z-1)^2}(1+x)\,,$$
$$\frac{\mathrm{d}z}{\mathrm{d}\ln a} = zx\,,$$

where the acceleration equation is

$$\frac{\dot{H}}{H^2} = -2 + 2\frac{z(z-1)}{\alpha+(z-1)^2}(1+x)\,,$$

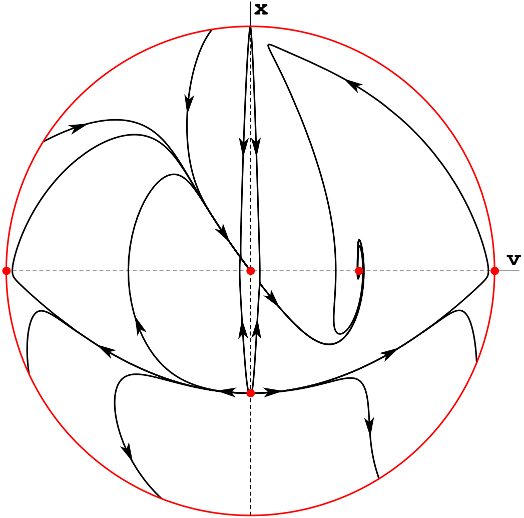
and the energy conservation condition

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\Lambda,0}}{\alpha}\frac{\alpha+(z-1)^2}{z(1+x)}\,,$$

dimensionless constant of the theory

$$\alpha = \frac{\Lambda}{\tilde{\Lambda}} = 8c_1\frac{\Lambda}{\phi_0}\,.$$

# $-2\phi_0\Lambda + \phi_0 R + c_1 R^2$ theory of gravity – the Jordan frame



# Into the Einstein frame

The Jordan frame action

$$S_g = \int \mathrm{d}^4x\sqrt{-g}\Big\{\varphi R - 2\phi_0\Lambda - \frac{(\varphi-\phi_0)^2}{4c_1}\Big\}$$

with the conformal factor

$$\Omega^2 = \frac{\varphi}{\phi_0}$$

transforms to the Einstein frame action integral

$$S_{\tilde{g}} = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\Big\{\phi_0\tilde{R} - \frac{3}{2}\frac{\phi_0}{\varphi^2}\tilde{\nabla}^\alpha\varphi\,\tilde{\nabla}_\alpha\varphi - 2\phi_0\frac{\phi_0^2}{\varphi^2}\left(\Lambda + \tilde{\Lambda}\left(\frac{\varphi}{\phi_0}-1\right)^2\right)\Big\}.$$

# The Einstein frame model

The geometry is given by the flat Friedmann-Robertson-Walker metric

$$d\tilde{s}^2 = \Omega^2 ds^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\Big(dx^2 + dy^2 + dz^2\Big)\,.$$

New dimensionless variables

$$\tilde{x} = \frac{\dot{\varphi}}{\tilde{H}\varphi}\,, \qquad z = \frac{1}{v} = \frac{\varphi}{\phi_0}\,.$$

The energy conservation condition is

$$\left(\frac{\tilde{H}}{\tilde{H}_0}\right)^2 = \frac{\tilde{\Omega}_{\Lambda,0} + \tilde{\Omega}_{\tilde{\Lambda},0}(z-1)^2}{z^2\left(1 - \frac{1}{4}\tilde{x}^2\right)}\,,$$

and the acceleration equation in the following form

$$\frac{\dot{\tilde{H}}}{\tilde{H}^2} = -\frac{3}{4}\tilde{x}^2\,,$$

where $\tilde{\Omega}_{\Lambda,0} = \frac{\Lambda}{3\tilde{H}_0^2}$, $\tilde{\Omega}_{\Lambda,0} = \frac{\tilde{\Lambda}}{3\tilde{H}_0^2}$ and $\tilde{\Lambda} = \frac{\phi_0}{8c_1}$ is the Einstein frame cosmological constant.

# The Einstein frame model – dynamics

The dynamical system

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\ln\tilde{a}} = 4\left(1-\frac{1}{4}\tilde{x}^2\right)\left(-\frac{3}{4}\tilde{x}+\frac{\alpha+1-z}{\alpha+(z-1)^2}\right),$$
$$\frac{\mathrm{d}z}{\mathrm{d}\ln\tilde{a}} = z\tilde{x},$$

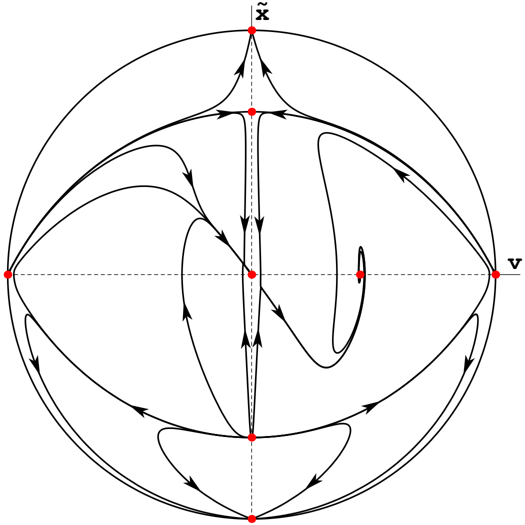
the energy conservation condition is

$$\left(\frac{\tilde{H}}{\tilde{H}_0}\right)^2 = \frac{\tilde{\Omega}_{\Lambda,0}}{\alpha}\frac{\alpha+(z-1)^2}{z^2\left(1-\frac{1}{4}\tilde{x}^2\right)},$$

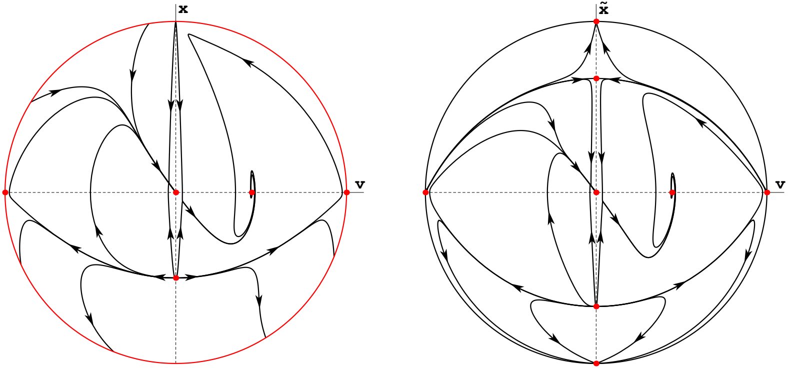
and dimensionless constant of the theory

$$\alpha = \frac{\Lambda}{\tilde{\Lambda}} = 8c_1\frac{\Lambda}{\phi_0}.$$

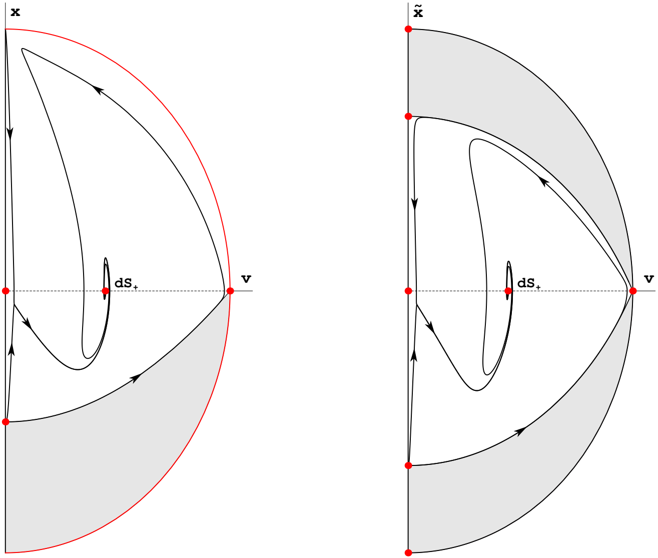
# $-2\phi_0\Lambda + \phi_0 R + c_1 R^2$ theory of gravity – the Einstein frame



# $-2\phi_0\Lambda + \phi_0 R + c_1 R^2$ – Jordan vs. Einstein



# $-2\phi_0\Lambda + \phi_0 R + c_1 R^2$ – Jordan vs. Einstein



The de Sitter state in both frames

$$x^* = \tilde{x}^* = 0\,, \quad z^* = 1+\alpha$$

the Jordan frame energy conservation condition gives

$$\left.\frac{H^2}{H_0^2}\right|^* = \Omega_{\Lambda,0} \quad \rightarrow \quad H^2\big|^* = \frac{\Lambda}{3}$$

while the Einstein frame energy conservation condition leads to

$$\left.\frac{\tilde{H}^2}{\tilde{H}_0^2}\right|^* = \tilde{\Omega}_{\Lambda,0}\frac{1}{1+\alpha} \quad \rightarrow \quad \tilde{H}^2\big|^* = \frac{\Lambda}{3}\frac{1}{1+\alpha}\,.$$

# The theory II

We start from the total action of the theory

$$S = S_g + S_\phi \,,$$

consisting of the gravitational part described by the standard Einstein-Hilbert action integral

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,R\,,$$

where $\kappa^2 = 8\pi G$, and the matter part of the theory is in the form of non-minimally coupled scalar field

$$S_\phi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big(\varepsilon\nabla^\alpha\phi\,\nabla_\alpha\phi + \varepsilon\xi R\phi^2 + 2U(\phi)\Big)\,,$$

where $\varepsilon = +1, -1$ corresponds to the canonical and the phantom scalar field, respectively.

# Into the Einstein frame

$$S^{(J)}_{g+\phi} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa^2}(1-\varepsilon\xi\kappa^2\phi^2)R - \varepsilon\nabla^\alpha\phi\nabla_\alpha\phi - 2U(\phi)\right)$$

conformal factor $\Omega^2 = |1-\varepsilon\xi\kappa^2\phi^2|$. The Jordan frame action integral transforms to the Einstein frame

$$S^{(J)}_{g+\phi} \to \mathrm{sign}\left(1-\varepsilon\xi\kappa^2\phi^2\right)\left(S^{(E)}_{\tilde{g}} + S^{(E)}_{\phi}\right)$$

with the gravitational part

$$S^{(E)}_{\tilde{g}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\tilde{R}$$

and the scalar field

$$S^{(E)}_{\phi} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left(\varepsilon\frac{1-\varepsilon\xi(1-6\xi)\kappa^2\phi^2}{(1-\varepsilon\xi\kappa^2\phi^2)^2}\tilde{\nabla}^\alpha\phi\,\tilde{\nabla}_\alpha\phi + 2\tilde{U}(\phi)\right)$$

where $\tilde{U}(\phi) = \frac{\mathrm{sign}\left(1-\varepsilon\xi\kappa^2\phi^2\right)}{(1-\varepsilon\xi\kappa^2\phi^2)^2}U(\phi)$.

# An asymptotically quadratic potential function

$$U(\phi) = \pm\frac{1}{2}m^2\phi^2 \pm M^{4+n}\phi^{-n}\,,$$

where $n > -2$
projective coordinates

$$u = \frac{\dot{\phi}}{H\phi}\,, \quad v = \frac{6}{\kappa^2\phi^2}\,,$$

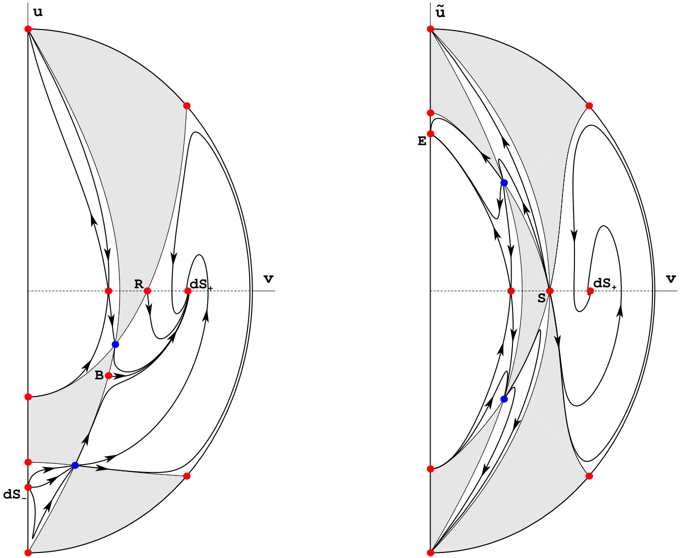
the energy conservation condition

$$\left(\frac{H}{H_0}\right)^2 = \frac{\mu + \alpha v^{1+\frac{n}{2}}}{v - \varepsilon(1-6\xi)u^2 - \varepsilon 6\xi(u+1)^2}$$

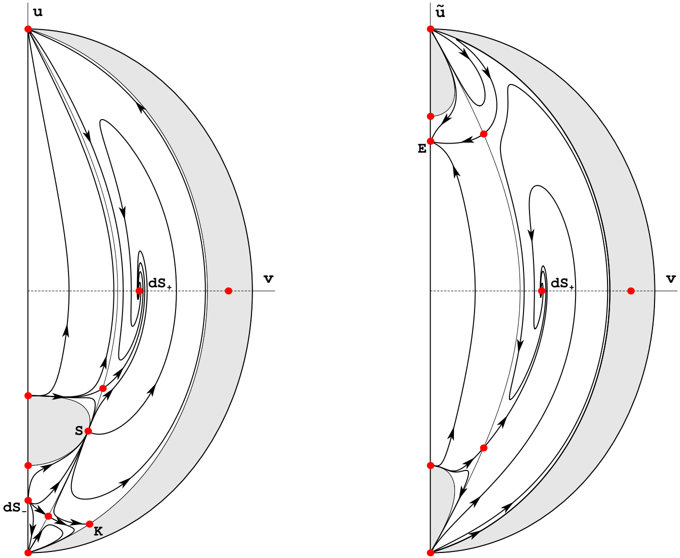
where

$$\mu = \pm\frac{m^2}{H_0^2}\,, \quad \alpha = \pm 2\left(\frac{\kappa^2}{6}\right)^{1+\frac{n}{2}}\frac{M^{4+n}}{H_0^2}\,,$$

# Non-minimally coupled scalar field



# Non-minimally coupled scalar field



# Conclusions

- We have shown differences between the Jordan frame and the Einstein frame formulations of two theories with geometrical and substantial origin of the primordial de Sitter evolution.
- In $-2\phi_0\Lambda + \phi_0 R + c_1 R^2$ theory of gravity in the Jordan and the Einstein frames there are different asymptotically stable de Sitter states.
- In the gravitational theory with non-minimally coupled scalar field and an asymptotically quadratic potential function with $\frac{3}{16} < \xi < \frac{1}{4}$ the expanding de Sitter solution in the Jordan frame is transformed into the Einstein static solution in the Einstein frame.
- The answer to the conundrum should be given by the agreement with observational data and the explanatory power of the theory under considerations (work in progress).

Or maybe some bold idea..