TT tensors in flat spaces of any dimension

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T and TT tensors

Condition

$$\nabla_i T^{ij} = 0$$

with symmetric T^{ij} appears in GR as

- the momentum constraint in the initial value problem
- gauge condition for the metric tensor
- "conservation law" of the energy-momentum.

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In flat spaces the TT conditions are solvable thanks to the Poincare lemma

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External indices can be added.

T tensors in flat spaces

Proposition

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$$T^{ij} = R^{ikjp}{}_{,kp}$$

where R^{ikjp} *is any tensor with algebraic symmetries of the Riemann tensor.*

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Symmetry $T^{ij} = T^{ji}$ yields

$$S^{[ij]k}{}_{,k} = 0$$

hence

$$S^{[ij]k} = V^{ijkp}{}_{,p}$$

and

$$V^{(ij)kp} = 0 = V^{ij(kp)} \ .$$

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Taking different permutations of *ijk* we obtain

$$S^{ijk} = (V^{ijkp} - V^{ikjp} - V^{jkip})_{,p}$$

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$$S^{ijk} = (V^{ijkp} - V^{ikjp} - V^{jkip})_{,p}$$

Let $R^{ijkp} = -V^{ikjp} - V^{jpik} + 2V^{[ikjp]}$

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D=2

$$T^{ij} = \epsilon^{ik}\epsilon^{jp}R_{,kp}$$

D=3

$$T^{ij} = -\epsilon^{ikl}\epsilon^{jps}G_{ls,kp}$$

where R and G_{ij} are, respectively, “the Ricci scalar” and “the Einstein tensor” corresponding to R^{ijkp}.

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In dimension D=2 the potential R is defined up to translation by $a_i x^i + b$.

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For $D \geq 3$ tensor R^{ikjp} is defined up to translation by

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Can the Weyl part or the Ricci part in the decomposition

$$R^{ik}{}_{jp} = C^{ik}{}_{jp} + 4a\delta^{[i}_{[j}R^{k]}_{p]} - 2bR\delta^{[i}_{[j}\delta^{k]}_{p]}$$

be eliminated?

Proposition

T tensors in $D = 3$ and all analytic T tensors in $D \geq 4$ are given by

$$T_{ij} = a \triangle R_{ij} - 2aR^{k}_{(i,j)k} + bR_{,ij} + (aR^{kp}{}_{,kp} - b \triangle R)g_{ij} \qquad (*)$$

where R_{ij} is a symmetric tensor undergoing gauge transformation

$$R_{ij} \longrightarrow R_{ij} + \xi_{(i,j)} - \xi^{k}{}_{,k} g_{ij}$$

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- Equations $(*)$ are linearized Einstein equations for the first corrections R_{ij} to the flat metric g_{ij}.
- Hence, the gauge freedom of R_{ij} is deduced (up to functions of D-1 variables).

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Proposition

In $D \geq 3$ TT tensors are given by

$$T^{ij} = R^{ikjp}{}_{,kp} \ ,$$

iff the "Ricci tensor" satisfies

$$R^{ij} = S^{(ij)k}{}_{,r} \ , \quad S^{ijk} = -S^{ikj} \ .$$

Gauge transformations of S^{ijk}:

$$S^{ijk} \longrightarrow S^{ijk} - 2g^{i[j}\xi^{k]} + \chi^{ijkr}{}_{,r} + \eta^{ijk} \ ,$$

where

$$\chi^{ijkr} = \chi^{i[jkr]} \ , \ \ \eta^{ijk} = \eta^{[ijk]} \ .$$

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Proposition

TT tensors in $D = 3$ are given by

$$T^{ij} = \epsilon^{kl(i}(\triangle A^{j)}{}_{k} - A_{kp,}{}^{j)p})_{,l}$$

where A_{ij} is a symmetric tensor defined up to the transformation

$$A_{ij} \longrightarrow A_{ij} + \chi_{(i,j)} + \eta g_{ij}$$

with arbitrary functions χ_i and η.

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Proposition

In dimension $D \geq 4$ every analytic TT tensor can be put into the form

$$T^{ij} = C^{ikjp}{}_{,kp}$$

where C^{ikjp} is a tensor with all symmetries of the Weyl tensor..

Perspectives

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  Not yet.

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- Description of symmetric TT tensors in flat spaces?
  Yes, but in D=3 the approach of Conboye and Ó Murchadha is better.
- Covariant description of TT tensors in conformally flat spaces?
  Not yet.
- Generalization to curved spaces?
  Only for spaces of constant curvature.