Black hole initial data within the parabolic-hyperbolic formulation

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Outline of the talk

Constraint equations

Hamiltonian constraint: $R + K^2 - K_{ij}K^{ij} = 0,$

momentum constraint: $D_j\left(K^{ij} - \gamma^{ij}K\right) = 0,$

where γ_{ij} is a metric on a 3-dim spacelike slice, $K_{ij} \equiv -\frac{1}{2}\mathscr{L}_n\gamma_{ij}$ is the extrinsic curvature with a trace K dubbed as the mean curvature

- underdetermined system – 4 equations for 12 independent elements of $\left(\gamma_{ij}, K_{ij}\right)$

The constraint equations bind the components of γ_{ij} and K_{ij} but they do not indicate which of them are free and which are constrained.

In the full nonlinear theory there is no unique decomposition of the components. Each decomposition yields a unique set of equations and a unique set of freely specifiable functions which need to be somehow fixed.

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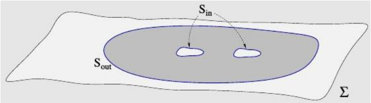
York-Lichnerowicz method & parabolic-hyperbolic formulation (also other...)

Black hole initial data within the York-Lichnerowicz conformal method

bh binary:



constraints $\longrightarrow$ elliptic equations

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boundary-value problem

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boundary data needed on outer **and inner** boundaries of the id region

Black hole initial data within the York-Lichnerowicz conformal method

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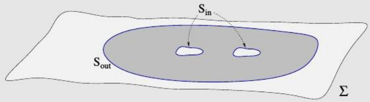

constraints $\longrightarrow$ elliptic equations

boundary-value problem

boundary data needed on outer **and inner** boundaries of the id region

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technicalities:
'constancy' of K^i_i or 'smallness' of T^{ij} required
implicitness due to conformal transformation (no control on physical parameters)
non-negligible spurious gw content produced during evolutions
problems with high black hole spins and boosts

2+1 split of Σ_0

Id surface is assumed to be foliated by a one-parameter family of codimension-one surfaces $\mathbb{S}_\rho$ with $\rho = const.$ being level surfaces of some smooth function $\rho : \Sigma_0 \to \mathbb{R}$

$$\Sigma_0 = \mathbb{R} \times \mathbb{S} \qquad \text{any foliation with } \mathbb{S} \text{ of arbitrary topology}$$

Decomposition of the 3-metric: $\gamma_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j$,

where $\hat{\gamma}_{ij}$ is the metric induced by γ_{ij} on the level surfaces $\mathbb{S}_\rho$ and $\hat{n}^i$ is the normal unit vector field

Decomposition of the extrinsic curvature: $K_{ij} = \boldsymbol{\kappa} \hat{n}_i \hat{n}_j + \hat{n}_i \mathbf{k}_j + \hat{n}_j \mathbf{k}_i + \mathbf{K}_{ij}$

where $\boldsymbol{\kappa} = \hat{n}^k \hat{n}^l K_{kl}$, $\mathbf{k}_i = \hat{\gamma}^k_l \hat{n}^l K_{kl}$, $\mathbf{K}_{ij} = \gamma^k_i \gamma^l_j K_{kl}$

A vector field $\rho^i = \widehat{\mathbb{N}} \hat{n}^i + \widehat{\mathbb{N}}^i$ can be introduced on Σ_0 s.t. $\rho^i D_i \rho = 1$ with the 'lapse' and 'shift' defined via

$$\hat{n}_i = \widehat{\mathbb{N}} \, (d\rho)_i \,, \qquad \widehat{\mathbb{N}}^i = \hat{\gamma}^i_j \rho^j .$$

Trace and tracefree parts of $\mathbf{K}_{ij}$: $\mathbf{K}^l_l = \hat{\gamma}^{kl} \mathbf{K}_{kl}$, $\overset{\circ}{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} \mathbf{K}^l_l$

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$$\left(\gamma_{ij}, K_{ij}\right) \Longleftrightarrow \left\{\widehat{\mathbb{N}}, \widehat{\mathbb{N}}^i, \hat{\gamma}_{ij}, \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l_l, \overset{\circ}{\mathbf{K}}_{ij}\right\}$$

Constraint equations in the parabolic-hyperbolic form

$$\overset{\star}{\mathbb{K}}\left(\partial_\rho \widehat{\mathbb{N}} - \widehat{\mathbb{N}}^l \hat{D}_l \widehat{\mathbb{N}}\right) - \widehat{\mathbb{N}}^2 \left(\hat{D}^l \hat{D}_l \widehat{\mathbb{N}}\right) - \mathscr{A}\widehat{\mathbb{N}} - \mathscr{B}\widehat{\mathbb{N}}^3 = 0$$

$$\mathscr{L}_{\hat{n}}\mathbf{k}_i - \tfrac{1}{2}\hat{D}_i \mathbf{K}^l{}_l - \hat{D}_i \boldsymbol{\kappa} + \hat{D}^l \overset{\circ}{\mathbf{K}}_{li} + \widehat{\mathbb{N}}^{-1}\overset{\star}{\mathbb{K}}\mathbf{k}_i + \left(\boldsymbol{\kappa} - \tfrac{1}{2}\mathbf{K}^l{}_l\right)\dot{\hat{n}}_i - \dot{\hat{n}}^l \overset{\circ}{\mathbf{K}}_{li} = 0$$

$$\mathscr{L}_{\hat{n}}\mathbf{K}^l{}_l - \hat{D}^l \mathbf{k}_l - \widehat{\mathbb{N}}^{-1}\overset{\star}{\mathbb{K}}\left(\boldsymbol{\kappa} - \tfrac{1}{2}\mathbf{K}^l{}_l\right) + \widehat{\mathbb{N}}^{-1}\overset{\circ}{\mathbf{K}}_{kl}\overset{\star}{\mathbb{K}}{}^{kl} + 2\dot{\hat{n}}^l \mathbf{k}_l = 0$$

where

$$\dot{\hat{n}}_k = \hat{n}^l D_l \hat{n}_k = -\hat{D}_k \ln \widehat{\mathbb{N}} \qquad \overset{\star}{\mathbb{K}}_{ij} = \tfrac{1}{2}\mathscr{L}_\rho \hat{\gamma}_{ij} - \hat{D}_{(i}\widehat{\mathbb{N}}_{j)} = \widehat{\mathbb{N}}\hat{K}_{ij}$$

$$\mathscr{A} = \partial_\rho \overset{\star}{\mathbb{K}} - \widehat{\mathbb{N}}^l \hat{D}_l \overset{\star}{\mathbb{K}} + \tfrac{1}{2}\left(\overset{\star}{\mathbb{K}}{}^2 + \overset{\star}{\mathbb{K}}_{kl}\overset{\star}{\mathbb{K}}{}^{kl}\right) \qquad \mathscr{B} = -\tfrac{1}{2}\left(\widehat{R} + 2\boldsymbol{\kappa}\mathbf{K}^l{}_l + \tfrac{1}{2}\mathbf{K}^l{}_l{}^2 - 2\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\overset{\circ}{\mathbf{K}}{}^{kl}\right)$$

$$\overset{\star}{\mathbb{K}} = \tfrac{1}{2}\hat{\gamma}^{ij}\mathscr{L}_\rho \hat{\gamma}_{ij} - \hat{D}_j \widehat{\mathbb{N}}^j = \widehat{\mathbb{N}}\hat{K}^l{}_l$$

$$\left(\gamma_{ij}, K_{ij}\right) \Longleftrightarrow \left\{\widehat{\mathbb{N}}, \widehat{\mathbb{N}}^i, \hat{\gamma}_{ij}, \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \overset{\circ}{\mathbf{K}}_{ij}\right\}$$

constrained: $\widehat{\mathbb{N}}, \mathbf{k}_i, \mathbf{K}^l{}_l$, **freely specifiable**: $\widehat{\mathbb{N}}^i, \hat{\gamma}_{ij}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{ij}$

Parabolic constraint

$$\overset{\star}{\mathbb{K}}\left(\partial_\rho\widehat{\mathbb{N}} - \widehat{\mathbb{N}}^l\hat{D}_l\widehat{\mathbb{N}}\right) - \widehat{\mathbb{N}}^2\left(\hat{D}^l\hat{D}_l\widehat{\mathbb{N}}\right) - \mathscr{A}\widehat{\mathbb{N}} - \mathscr{B}\widehat{\mathbb{N}}^3 = 0$$

where

$$\mathscr{A} = \partial_\rho\overset{\star}{\mathbb{K}} - \widehat{\mathbb{N}}^l\hat{D}_l\overset{\star}{\mathbb{K}} + \tfrac{1}{2}\left(\overset{\star}{\mathbb{K}}{}^2 + \overset{\star}{\mathbb{K}}_{kl}\overset{\star}{\mathbb{K}}{}^{kl}\right) \qquad \mathscr{B} = -\tfrac{1}{2}\left(\widehat{R} + 2\kappa\mathbf{K}^l{}_l + \tfrac{1}{2}\mathbf{K}^l{}_l{}^2 - 2\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\overset{\circ}{\mathbf{K}}{}^{kl}\right)$$

$$\overset{\star}{\mathbb{K}} = \tfrac{1}{2}\hat{\gamma}^{ij}\mathscr{L}_\rho\hat{\gamma}_{ij} - \hat{D}_j\widehat{\mathbb{N}}^j = \widehat{\mathbb{N}}\hat{K}^l{}_l$$

- uniformly parabolic in subregions of Σ_0, where $\overset{\star}{\mathbb{K}}$ is either positive or negative
- $\overset{\star}{\mathbb{K}}$ depends only on the freely specifiable fields
- $\overset{\star}{\mathbb{K}}$ determines the integration step in the 'temporal' direction, $h_\rho \leqslant \frac{1}{2}\overset{\star}{\mathbb{K}}\widehat{\mathbb{N}}^{-1}\left(\min h_{non-'temporal'}\right)^2$

Kerr-Schild type of a binary bh metric

Superposed Kerr-Schild black holes metric is

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2H^{[1]} l_\alpha^{[1]} l_\beta^{[1]} + 2H^{[2]} l_\alpha^{[2]} l_\beta^{[2]} \qquad (^*)$$

where the Kerr-Schild data of individual bhs are the following: H is a smooth function on $\mathbb{R}^4$ apart from singularity and l_α is null with respect to both $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$, which for a single Kerr bh in the Cartesian coordinates are

$$H = \frac{r^3 M}{r^4 + a^2 z^2}, \qquad l_\alpha = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$

with the Boyer-Lindquist radial coordinate satisfying $r^4 - \left(x^2 + y^2 + z^2 - a^2\right) r^2 - a^2 z^2 = 0.$

$(^*)$ does not solve the Einstein eqns !

Choice of the initial-boundary data for solving the constraints:

- freely specifiable: $\widehat{\mathbb{N}}^i, \hat{\gamma}_{ij}, \boldsymbol{\kappa}, \overset{\circ}{\mathbf{K}}_{ij}$ as if $(^*)$ was a solution to the Einstein eqns
- constrained: $\widehat{\mathbb{N}}, \mathbf{k}_i, \mathbf{K}^l{}_l$ imposed according to $(^*)$ only on a boundary

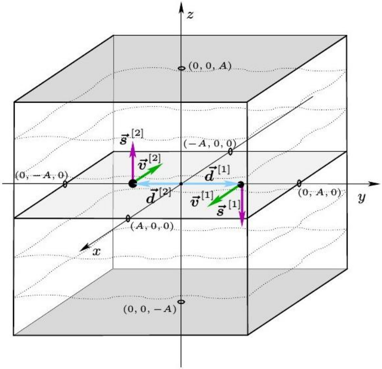
$$\Downarrow$$

$$\left\{\widehat{\mathbb{N}}, \mathbb{k}_1, \mathbb{k}_2, \mathbb{K}\right\}$$

Practice: numerical computations

- Σ_0 is chosen to be large (in comparison to bh masses/radii) bounded subset of $\mathbb{R}^3$
- the structure $\Sigma_0 = \mathbb{R} \times \mathbb{S}$ can be realized by $\Sigma_0 = \mathbb{R}^3$ with $\mathbb{S}_\rho$ leaves being $\mathbb{R}^2$

Assumed arrangement of the binary bh system:



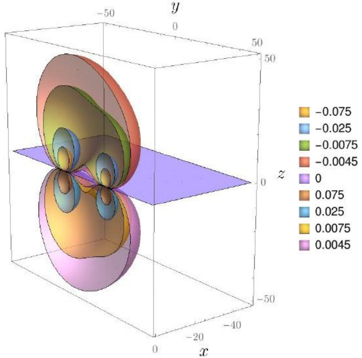
Input parameters:
rest masses M, displacements d,
speeds/boosts v and spins a
of individual black holes

- bhs assumed to be located on the $z = 0$ plane
- boosts are parallel, spins are orthogonal to $z = 0$
- foliation by $z = const.$ level surfaces

Practice: numerical computations

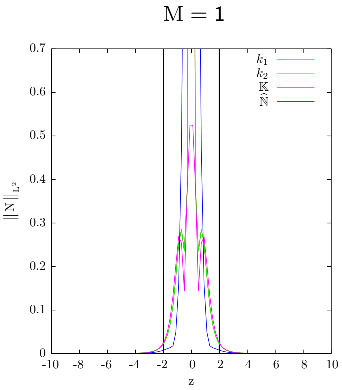
- the principal coefficient $\overset{\star}{\mathbb{K}} \overset{!}{=} z \cdot f^{<0}$ is positive for $z < 0$ and negative for $z > 0$ (decides whether the system evolves in positive or negative direction of $\rho \equiv z$),

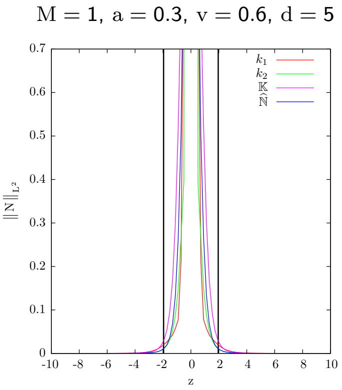
e.g., for $M_1 = 1$, $d_1 = 20$, $v_1 = 0.5$, $a_1 = 0.6$,, $M_2 = 2$, $d_2 = 10$, $v_2 = -0.25$, $a_2 = -0.8$



forces an adaptive z-step during integration

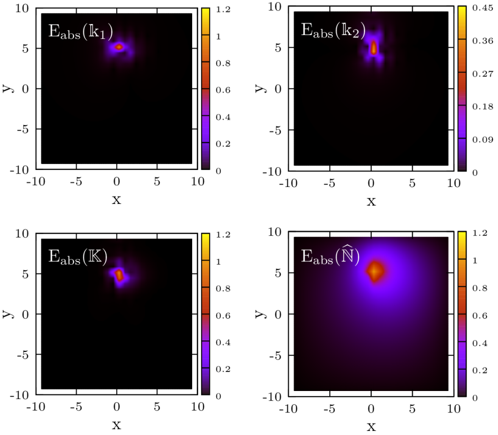
L^2 norms vs. z: $\|\mathrm{N}\|_{\mathrm{L}^2} = \sqrt{\sum_i \sum_j |\mathrm{N}_{ij}|^2}$, where $\mathrm{N} = F - F^{analytic}$, $F = \{\Bbbk_1, \Bbbk_2, \mathbb{K}, \widehat{\mathbb{N}}\}$





Absolute errors: $E_{abs}(F) = \left|F - F^{analytic}\right|$ on $z = 0.25 r_H$

$\mathrm{M} = 1,\ \mathrm{a} = 0.3,\ \mathrm{v} = 0.6,\ \mathrm{d} = 5$

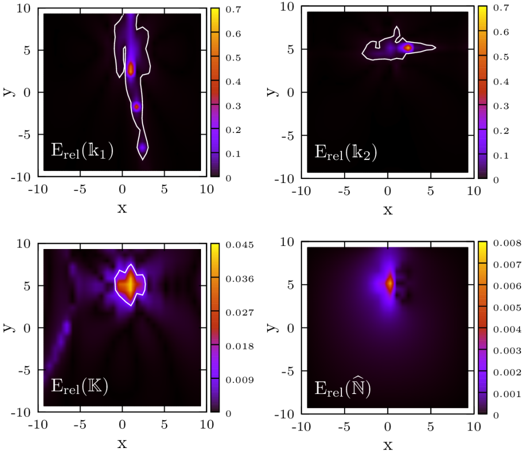


overall accuracy of the numerical method: $F^{analytic} = F + O\left(h^n\right)$,

where $h \equiv \max(h_x, h_y, h_z)$ and $n = 4$

Relative errors: $E_{rel}(F) = \left|F - F^{analytic}\right|\left|F^{analytic}\right|^{-1}$ for $\mathrm{M} = 1,\ \mathrm{a} = 0.3,\ \mathrm{v} = 0.6,\ \mathrm{d} = 5$

$z = r_H$



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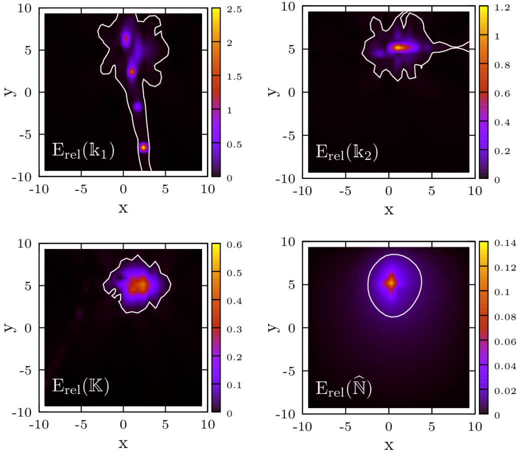
$z = 0.5 r_H$



Deviations – another form of the constrained functions, $F^{\Delta} = \{\Delta \mathbb{k}_1, \Delta \mathbb{k}_2, \Delta \mathbb{K}, \Delta \widehat{\mathbb{N}}\}$

$$\Delta \widehat{\mathbb{N}} = \widehat{\mathbb{N}} - {}^{(A)}\widehat{\mathbb{N}}, \quad \Delta \mathbb{K} = \mathbb{K} - {}^{(A)}\mathbb{K}, \quad \Delta \mathbb{k} = \mathbb{k} - {}^{(A)}\mathbb{k},$$

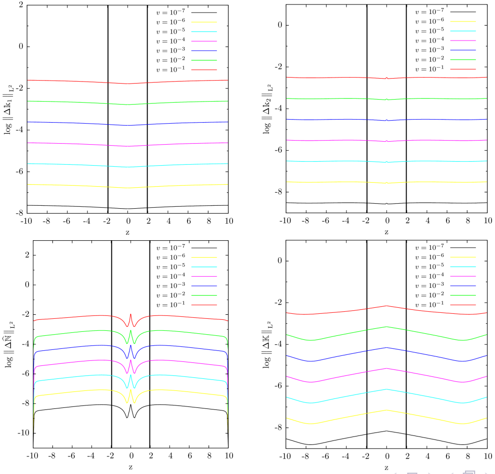
where ${}^{(A)}\widehat{\mathbb{N}}$, ${}^{(A)}\mathbb{K}$ and ${}^{(A)}\mathbb{k}$ are analytic background functions, for which the most natural choice is the one related to an appropriate Kerr-Schild metric form

- equal to zero everywhere for single black holes
- distorted black holes – background functions of a single black hole, initial-boundary data for the same one, slightly boosted or displaced

Deviations – another form of the constrained functions, $F^{\Delta} = \{\Delta \mathbb{k}_1, \Delta \mathbb{k}_2, \Delta \mathbb{K}, \Delta \widehat{\mathbb{N}}\}$

Distorted Kerr black hole: logs of L^2 norms of deviations vs. *z* for various boosts v.

The background: M=1 and a=0.4. IB data: a boosted black hole with the same M and a.

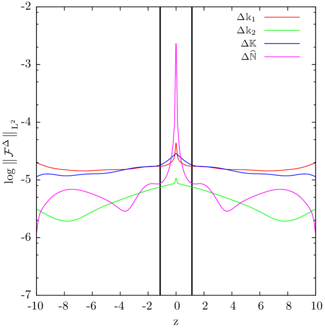


Highly boosted and highly spinning black holes

Distorted highly boosted and highly spinning black hole: logs of $\mathbf{L}^2$ norms of deviations vs. z.

The background: M=1, a=0.99 and v=0.99.

IB data: a displaced black hole with the same M, a and v, with d= 10^{-4}M.



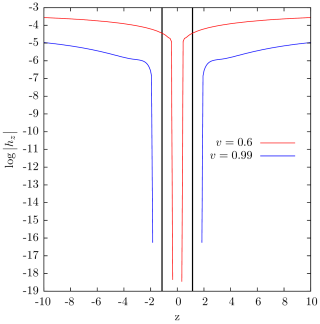
Highly boosted and highly spinning black holes

Log of an adaptive *z*-step vs. *z*.

The full functional formulation of the constraints – a highly boosted, highly spinning and displaced Kerr black hole with M=1, a=0.99M, v=0.99, d=3M

and a black hole characterized by the same M, a and d

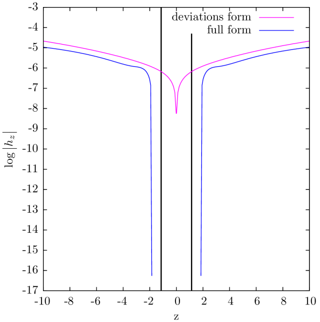
but a smaller boost v=0.6.



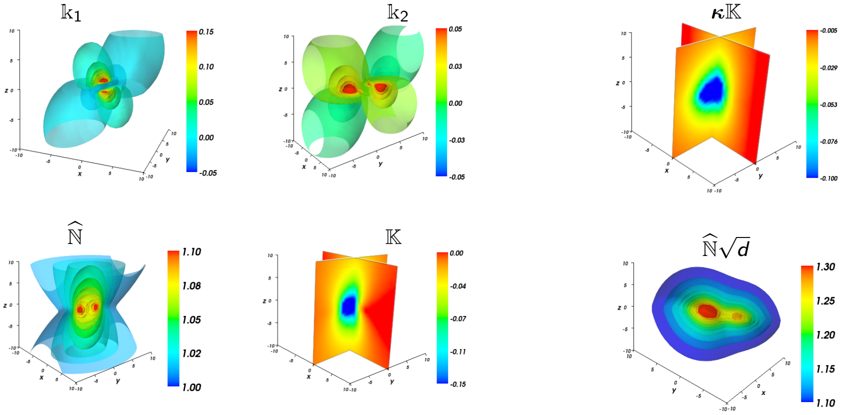
A highly boosted, highly spinning and displaced Kerr black hole with M=1, a=0.99M, v=0.99

in the deviations formulation of the constraints with d= 10^{-4}M

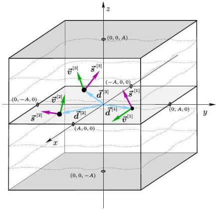
and their full functional form with d=3M.



$\mathrm{M}_1 = 0.2, \mathrm{a}_1 = 0.3, \mathrm{v}_1 = 0.6, \mathrm{d}_1 = 2,$ $\mathrm{M}_2 = 0.2, \mathrm{a}_2 = -0.4, \mathrm{v}_2 = 0.7, \mathrm{d}_2 = 2$



A new method of formulating initial data of the Einstein equations which involve multiple black holes was proposed, tested and used for a black hole systems initial description.



- The method has an 'evolutionary' character as it is based on solving a coupled system of one parabolic and three hyperbolic equations.
- The method does not require imposing boundary conditions in the strong field regime, which is a big advantage over the traditional conformal method.
- There are no restrictions on the bh parameters (apart from the obvious ones).
- The numerical code prepared for solving the system of equations of the initial data in the case of single bhs is highly accurate within the whole computational domain when compared with the analytic result.
- Work in progress: horizon search and black hole parameters computation within the parabolic-hyperbolic approach, time evolution of the initial data.

Bibliography

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THANK YOU FOR YOUR ATTENTION.